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The Cauchy Problem in Spacetimes with Closed Timelike Curves

by

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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The Cauchy problem for the scalar wave equation is studied in spacetimes with closed timelike curves. In general such spacetimes contain a chronal and non-chronal region, separated by a chronology horizon. We consider spacetimes that contain compactly generated horizons and those that do not.

Initially we consider initial data for the wave equation on partial Cauchy surfaces in the chronal region. In Misner space we can write down an explicit solution to the wave equation in terms of the initial data. It is shown that generic initial data evolves to give a divergent stress-energy scalar at the chronology horizon. The Cauchy problem in the wormhole spacetimes is discussed in the geometric optics limit. We find that initial data can evolve in a finite way through the chronology horizon. Using a covering space we were able to convert the Cauchy problem in Gott space to an equivalent problem in Minkowski space. Modifying results in Hawking and Ellis we show that the Cauchy problem in Gott space is well posed up to and on the chronology

horizon.

We then considered extending the data for the wave equation beyond the chronology horizon in Gott space and Grant space. Performing a change of coordinates so that the isometries of the spacetime are manifest in one periodic coordinate we simplify the wave equation to a reduced wave equation. This is an equation of mixed type, changing from elliptic to hyperbolic. Similar things happen when we consider the wave equation in the spinning cosmic string spacetime. The surface of parabolic degeneracy coincides with the "0^{th}-polarised hypersurface". The nature of the "0^{th}-polarised hypersurface" is discussed and we give a definition of this surface that is not dependent on the spacetime having isometries. We define a surface called "the essential chronology horizon" which we conjecture coincides with the 0^{th}-polarised hypersurface for spacetimes without compactly generated chronology horizon and the chronology horizon for spacetimes with compactly generated chronology horizon.

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Chapter 1

INTRODUCTION

1.1 Background

Gödel first showed in 1949 that it is possible to construct a time machine using Einstein's general theory of relativity. Stated more precisely, Gödel discovered an exact solution to the Einstein field equations that contained closed timelike curves (CTCs). A CTC is a path in spacetime such that any material particle following the CTC, while restricted to its own forward light cone, eventually returns to the same point in space *and* time from which it started.

The are many other solutions to Einstein's field equations that contain CTCs, such as the Kerr black hole solution, the Taub-NUT solution and the van-Stockum solution. Many of these solutions can be ruled out on physical grounds, for example Gödel's solution is a cosmological solution with CTCs through every point in the spacetime. From everyday experience it is easy to see that our universe does not contain CTCs everywhere (it is still possible, however, that Gödel's solution could describe part of our universe). Recently there has been renewed interest in the study of CTCs due to two new solutions that could not be so readily dismissed on physical grounds (even if they are based on slightly hypothetical objects). The first of these being CTCs created from moving wormholes [MTY,FMN] and the second being CTCs created from rapidly moving cosmic strings [G].

Naively one may dismiss CTCs straight away as it is well known that many paradoxical situations could arise if one had access to a time machine (such as, for example, the possibility of going back in time and killing ones parents). Do CTCs present real problems for physics (and if so is there some physical mechanism by which they can be ruled out), or are CTCs not as problematic as they may at first seem.

One way to investigate the *nastiness* of CTCs is to study the Cauchy problem in such spacetimes. The spacetimes we will concentrate on here all share the feature that they contain an initially globally hyperbolic region that evolves into a timemachine. The causal and non causal regions are separated by a chronology horizon. If CTCs are permitted by the laws of physics, and if CTCs are to leave the laws of physics self consistent, it seems reasonable to hope that such spacetimes have a well defined Cauchy problem. That is, initial data (for example a scalar field) on a partial Cauchy surface preceding the Chronology horizon will evolve in a unique, finite and continuous way up to and beyond the chronology horizon. Here we will consider the Cauchy problems for Misner space, Thorne's wormhole spacetime, Gott space and the spinning cosmic string spacetime. In the next chapter we will give a detailed description of the aforementioned spacetimes, and in following chapters we shall investigate the Cauchy problems on these spacetimes.

Chapter 2

THE SPACETIMES

2.1 Misner space

The spacetimes of main interest in this thesis can be thought of as Minkowski space with various exotic identifications made. The resulting non-trivial global structure of the spacetime is responsible for the closed timelike curves. Misner space is an example of such a spacetime and provides a useful test bed for ideas concerning some of the more complicated examples such as Grant space, Gott space and various wormhole spacetimes considered later.

We will consider 2−dimensional Misner space with cylindrical topology $R^1 \times S^1$ and metric given by

$$ds^2 = -T^{-1}dT^2 + TdX^2, \tag{2.1}$$

where $-\infty < T < 0$ and $0 \leq X \leq 2\pi$.

Now the above metric (2.1) is singular at $T = 0$, but we can extend across the singularity by making the following coordinate change

$$X' = X - \ln T \tag{2.2}$$

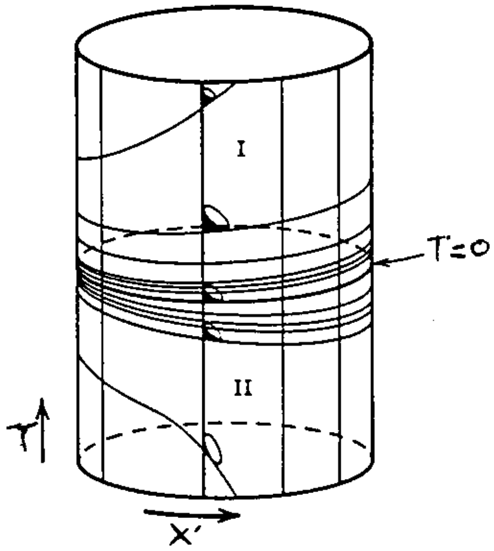

Figure 2.1: *Extension of Misner space across* $T = 0$

This will now take our original spacetime $(M, \mathbf{g})$ onto $(M', \mathbf{g}')$, where $\mathbf{g}'$ is now defined by

$$ds^2 = 2dX'dT + TdX'^2. \tag{2.3}$$

The new manifold is now free of singularities and is defined for $-\infty < T < \infty$. The region $T < 0$ of M' is isometric to M, but the region $T > 0$ contains CTCs. There are two sets of null geodesics: one set crosses the surface $T = 0$, while the other set spiral round and round as they approach $T = 0$, but never cross $T = 0$. The vertical set of geodesics are complete and extend from $T = -\infty$ to $T = \infty$. The spiralling geodesics are incomplete.

Alternatively we could have extended $(M, \mathbf{g})$ to $(M'', \mathbf{g}'')$ by putting $X'' = X + \ln T$ giving the metric:

$$ds^2 = -2dX''dT + TdX''^2. \tag{2.4}$$

This simply has the effect of interchanging the roles of the two sets of null geodesics.

One can see that we have two inequivalent extensions of $(M, \mathbf{g})$ both of which are geodesically incomplete.

At $T = 0$ we have a closed null geodesic. A closed null geodesic is a null geodesic that starts at event P and eventually returns to event P with the same tangent direction.

It can be seen that for $T > 0$ the light cones are tipped sufficiently so as to allow CTCs to form. This is not the case for $T < 0$. The surface $T = 0$ separates the causal and non causal regions and is called the chronology horizon.

The properties of Misner space can be made clear by going to the covering space $(\tilde{M}, \tilde{\mathbf{g}})$. Here $\tilde{\mathbf{g}}$ is given by (2.1), but $-\infty < X < \infty$ and is not periodic. We now make the coordinate transformation to (t, x) coordinates given by

$$\begin{aligned} t &= 2\sqrt{-T}\cosh\frac{X}{2}, \\ x &= 2\sqrt{-T}\sinh\frac{X}{2}, \end{aligned}$$

where $0 < t < \infty$, $-\infty < x < \infty$ with $t^2 > x^2$, with inverse

$$\begin{aligned} T &= \frac{1}{4}(x^2 - t^2), \\ X &= 2\tanh^{-1}\frac{x}{t}. \end{aligned}$$

In these coordinates the metric $\tilde{\mathbf{g}}$ becomes

$$ds^2 = dt^2 - dx^2. \qquad (2.5)$$

Hence the covering space $(\tilde{M}, \tilde{\mathbf{g}})$ is just $(I, \tilde{\nu})$ where region I of Minkowski space is contained within the future null cone of the point p (see figure (2.2)). To obtain Misner space we must identify the point (T, X) with the point $(T, X + 2\pi)$. In the (t, x) coordinates this is equivalent to identifying (t_0, x_0) with the point

$$(t_1, x_1) = (t_0 \cosh \pi + x_0 \sinh \pi, x_0 \cosh \pi + t_0 \sinh \pi).$$

The transformation A which maps (t_0, x_0) to (t_1, x_1) is simply a boost in the two-dimensional Minkowski space which leaves the future null cone of p invariant. The orbits being points lying on the hyperbolae $t^2 - x^2 = K$, where K is a constant. Thus $(M, \mathbf{g})$ may be obtained from $(I, \tilde{\nu})$ by quotienting out by the action of G, where G is the group of isometries generated by A.

One can extend this covering into region II of $(\tilde{M}, \tilde{\nu})$. The quotient $((I+II), \tilde{\nu})/G$ is now the Hausdorff space $(M, \mathbf{g}')$. Thus the covering space of the extended Misner space $(M, \mathbf{g}')$ is equivalent to the regions I and II of Minkowski space with points identified under a boost A^n defined above. One can see from the covering space how one set of null geodesics can be completed. Consider figure (2.2), the null geodesics going from the bottom right to the top left in region $I + II$ correspond to the vertical geodesics intersecting the chronology horizon in $(M, \mathbf{g}')$. These geodesics are incomplete. The geodesics going from the top-right to the bottom-left in regions I and II correspond to the incomplete spiralling geodesics in $(M, \mathbf{g}')$.

Alternatively one could have extended the covering into region III of $(\tilde{m}, \tilde{\nu})$. The quotient $((I + III), \tilde{\nu})/G$ is now the Hausdorff space $(M'', \mathbf{g}'')$. One can see how the roles of the two sets of null geodesic are swapped by going between the two different coverings.

One might think that we perform both extensions simultaneously. However the quotient $(I + II + III, \tilde{\nu})/G$ is not a Hausdorff manifold.

It is possible to do similar extensions and consider the whole of two dimensional Minkowski space $(\tilde{m}, \tilde{\nu})$ identified under the action of the group G. However in order for the resulting space to be a manifold one must delete the origin. The resulting manifold $(\tilde{M} - p, \tilde{\nu})$ is non-Hausdorff and the space is geodesically incomplete because of the missing point p.

To get a more physical idea of what Misner space is consider the following. Walk into a room, identify the front wall with the back wall (in such a fashion so that when you walk into the back wall you reappear coming out of the front wall), identify the left wall with the right wall and identify the floor with the ceiling. Now start the left wall moving towards the right wall and you get Misner space.

In the covering space $(\tilde{m}, \tilde{\nu})$ the chronology horizon is now at $X^2 - T^2 = 0$, with causality violation occurring in the region $X^2 - T^2 > 0$. There is a closed null geodesic at $X^2 - T^2 = 0$.

2.2 Grant Space

Grant space [Gr] is a generalisation of Misner space. As with Misner space, the covering space for Grant space is Minkowski space $(\tilde{m}, \tilde{\nu})$. The space is then obtained

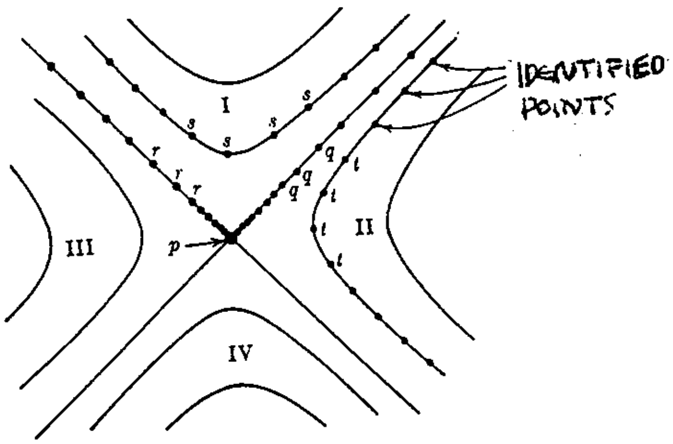

Figure 2.2: *The Covering space for Misner space The causality violation occurs in region II. Under the actions of the group G the points marked q are equivalent, as are the one marked r, s and t.*

by identifying points under the action of a discrete group of isometries H. An element B of the group H maps a point (t, x, y, z) to the point

$$(t \cosh 4a + x \sinh 4a, x \cosh 4a + t \sinh 4a, y + 4d, z). \tag{2.6}$$

So one identifies points of the form

$$(t \cosh 4na + x \sinh 4na, x \cosh 4na + t \sinh 4na, y + 4nd, z), \tag{2.7}$$

for all $n \in \mathbf{Z}$. So as well points being identified under a boost in the $t - x$ plane, there is an extra translation in the y direction. The resulting quotient space is a Hausdorff manifold (unless $d = 0$) and the space is geodesically complete. Grant space is therefore the quotient $(\tilde{m}, \tilde{\nu})/H$.

A discussion on the causal structure of Grant space will be given later on in the section on Gott space.

2.3 Gott Space

Gott space is a solution of Einstein's equations representing two parallel moving cosmic strings of infinite length (for more detail see [G,C]). A cosmic string is an object that is thought to form as a result of phase transition in the early universe. It has exterior metric

$$ds^2 = dt^2 - dr^2 - (1-4\mu)^2 r^2 d\phi^2 - dz^2, \tag{2.8}$$

where $-\infty < t < \infty$, $-\infty < z < \infty$, $0 < r < \infty$ (for infinitely thin strings). μ is the mass per unit length of the cosmic string, with typical value $\mu \simeq 10^{-6}$ in natural units . As the structure of the cosmic string spacetime is uniform in the z direction we will just consider the t, r and θ directions.

A $t =$ *constant* slice of the cosmic string spacetime has conical geometry. This can be made clear by defining

$$\phi' = (1-4\mu)\phi, \tag{2.9}$$

with $0 \leq \phi' \leq (1-4\mu)2\pi$. The metric then becomes (neglecting z)

$$ds^2 = dt^2 - dr^2 - r^2 - d\theta'^2. \tag{2.10}$$

We therefore have Minkowski space with a wedge of angle $8\pi\mu$ missing, and the events $(t, r, \phi' = 0)$ and $(t, r, \phi' = 2\pi - 8\pi\mu)$ identified.

From now on we will consider only cosmic strings with deficit angle π. These spacetimes contain all the important features of spacetimes with deficit angle less

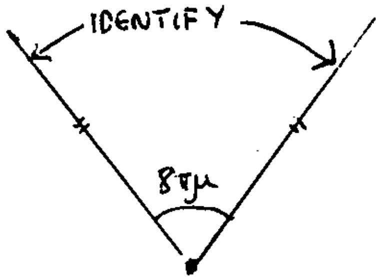

Figure 2.3: *The wedge removed from Minkowski space. Events of either side of the wedge are identified.*

than π, and provide a useful symmetry that will be of use later when it come to constructing a covering space for Gott space. For a discussion of Gott space and its global structure with general angular deficit see [G,C].

Constructing the two string solution is a simple matter of gluing two cosmic strings together along the three-surface $y = 0$ (see figure (2.4) . We can perform this gluing as the three-surface $y = 0$ is intrinsically and extrinsically flat.

The resulting two string spacetime is a region of Minkowski space $(-d \leq y \leq d)$ with the events $(t, x, y = d)$ and $(t, -x, y = d)$ identified, and the events $(t, x, y = -d)$ and $(t, -x, y = -d)$ identified.

Events occurring on opposite sides of the wedge are simultaneous for observers at rest with respect to the strings.

Now consider figure (2.4), an observer at event B; he will see three images of an object at A. There are three ways for light to travel from A to B: directly through the origin, and indirectly via the wedges. Note that if the cosmic strings were not

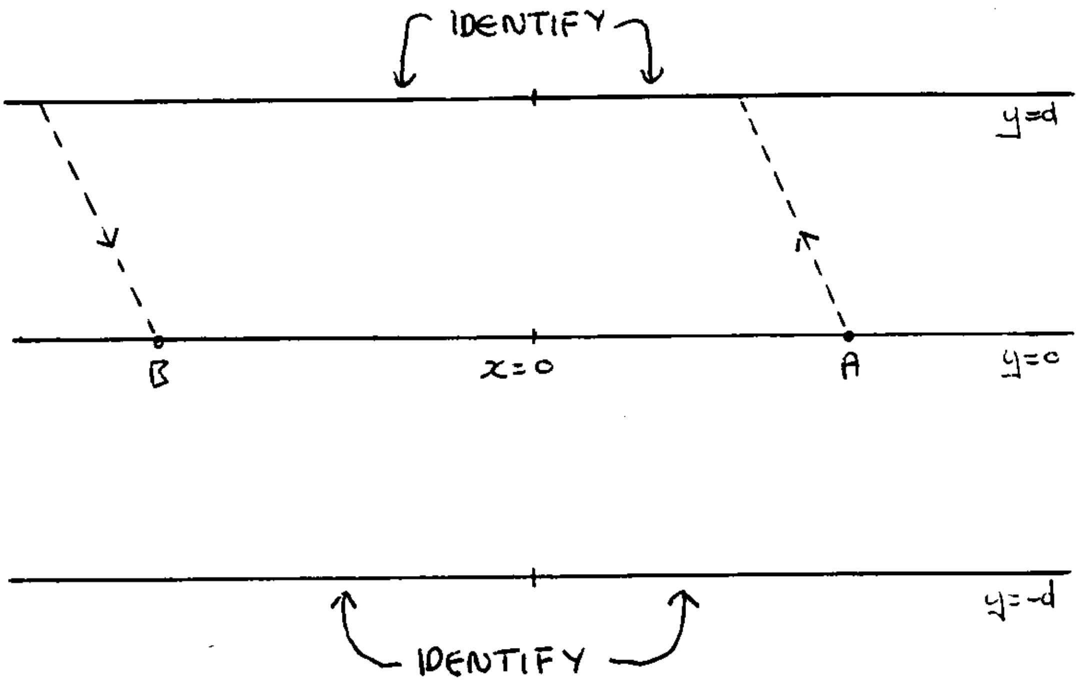

Figure 2.4: *The two string spacetime. The dotted line depicts a null geodesic going from A to B via the string.*

present then the quickest route for the light to go would be via the origin.

Now consider a rocket traversing this spacetime. In order to get CTCs, as will be explained later, we would like our rocket to go from A to B faster than a light beam can go from A to B via the origin. Obviously we would like the rocket to take the shortest route possible. That route being travelling vertically upwards from A to the string wedge, going via the wedge and then flying down to B (see figure (2.4)).

If the rocket travels with velocity v_r then events A and B are given by the following

$$A = (-v_r^{-1}d, x_0, 0), \tag{2.11}$$

$$B = (v_r^{-1}d, -x_0, 0). \tag{2.12}$$

A and B are spacelike separated in the surface $y = 0$ (the rocket going via the string beats the light ray going via the origin) if

$$v_r^{-2} d^2 < x_0^2, \tag{2.13}$$

that is if $v_r > d x_0^{-1}$. For $v_r < 1$ we require $x_0 > d$.

So provided $x_0 > d$ we can use our rocket to join two events which are spacelike separated in the space given by $y = 0$. As A and B are spacelike separated (in $y = 0$) there exists a Lorentz frame in which they are simultaneous. Now take the string solution for $y \geq 0$ and give it velocity v_s in the positive x direction. In the laboratory (centre of mass) frame the events A and B are now given by

$$A = \gamma(-v_r^{-1} d + v_s x_0, x_0 - v_s v_r^{-1} d), \tag{2.14}$$

$$B = \gamma(v_r^{-1} d - v_s x_0, -x_0 + v_s v_r^{-1} d), \tag{2.15}$$

where $\gamma = (1 - v_s^2)^{1/2}$.

Now provided $v_s = v_r^{-1} d x_0^{-1} < 1$ the events A and B are simultaneous in the laboratory frame.

Similarly take the solution for $y \leq 0$ and give it velocity v_s in the $-x$ direction. With $v_s = v_r^{-1} d x_0^{-1}$ the events A and B take the following form in the laboratory frame

$$A = (0, \gamma(x_0 - v_r^2 d^2 x_0^{-1})), \tag{2.16}$$

$$B = (0, \gamma(-x_0 + v_r^2 d^2 x_0^{-1})). \tag{2.17}$$

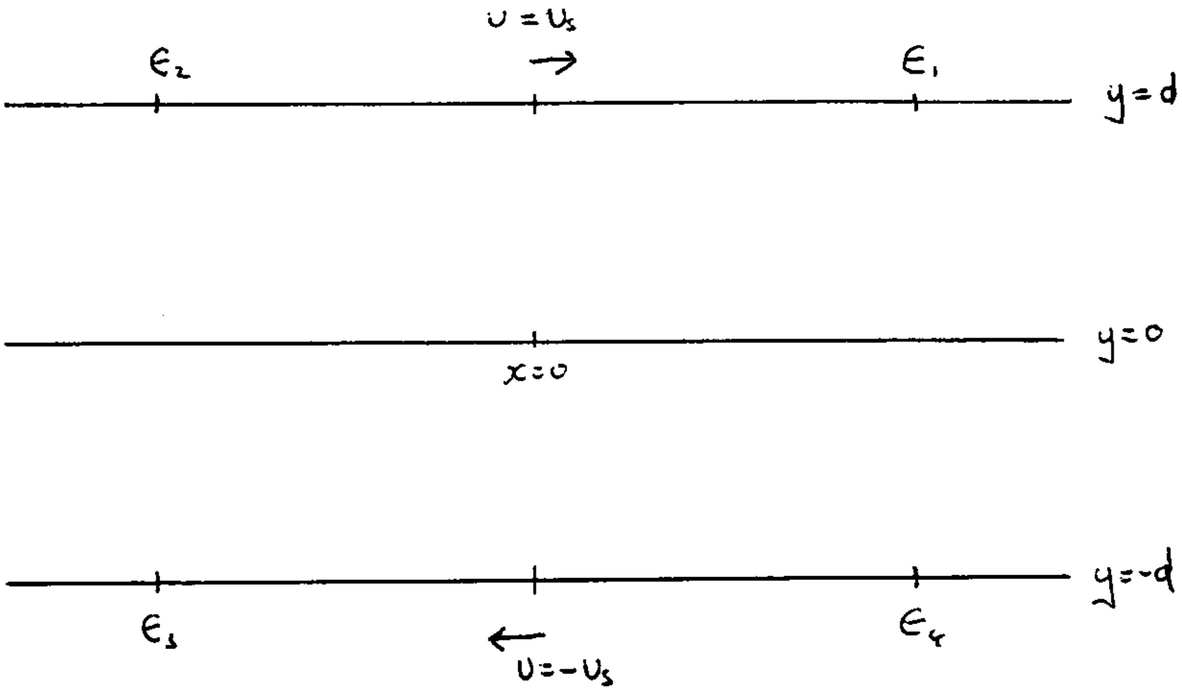

Figure 2.5: *The Gott spacetime.*

It should be noted that the boosts applied do not change the nature of the surface $y = 0$, and the two solutions can still be glued together.

We are now in a position to travel along a CTC. Take the top string and move it with velocity v_s in the $+x$ direction. Start at event A and travel, via the top string, to event B which is simultaneous with A in the laboratory frame. Now move the bottom string in the $-x$ direction. Proceed from event B to event A, which is simultaneous with event B in the laboratory frame. We have now traversed a closed loop in space and time—coming back to when and where we started.

There is an alternative way of considering how CTCs arise. Consider the original static solution and the events given below

$$E_1 = (t, x_0, d), \tag{2.18}$$

$$E_2 = (t, -x_0, d), \tag{2.19}$$

$$E_3 = (t, -x_0, -d), \tag{2.20}$$

$$E_4 = (t, x_0, -d). \tag{2.21}$$

where E_1 and E_2 are identified, and E_3 and E_4 are identified (see fig(2.5))

Now boost the top string in the $+x$ direction, and the bottom string in the $-x$ direction with velocity v_s. The events E_1, E_2, E_3 and E_4 now become

$$E_1 = (\gamma(t + v_s x_0), \gamma(x_0 + v_s t), d), \tag{2.22}$$

$$E_2 = (\gamma(t - v_s x_0), \gamma(-x_0 + v_s t), d), \tag{2.23}$$

$$E_3 = (\gamma(t + v_s x_0), \gamma(-x_0 - v_s t), -d), \tag{2.24}$$

$$E_4 = (\gamma(t - v_s x_0), \gamma(x_0 - v_s t), -d). \tag{2.25}$$

in the laboratory frame. Where the events E_1 and E_2 are identified, and E_3 and E_4 are identified.

Now to get a CTC here we want to arrange for v_s to allow the following: start at E_2, fly down the y-axis such that when we arrive at $y = -d$ we coincide with E_3. E_4 is identified with E_3, so we now proceed from E_4 and fly up the y-axis, arriving at $y = d$ so as to coincide with E_1. E_1 is identified with E_2, so we have traversed a CTC. We have used the above identifications to "travel back in time" twice.

Now in order to complete this CTC we require that the distance covered in going from E_2 to E_3 (and E_4 to E_1), which is exactly $\sqrt{(2d)^2 + (2v_s\gamma t)^2}$, be the same as the distance travelled by the rocket.

Taking v_r to be the velocity of the rocket we want

$$v_s v_r x_0 \gamma = \sqrt{d^2 + \gamma^2 t^2 v_s^2}. \tag{2.26}$$

Now our rocket cannot travel faster than light so

$$v_s x_0 \gamma > \sqrt{d^2 + \gamma^2 t^2 v_s^2}, \tag{2.27}$$

recall $\gamma^{-1} = \sqrt{1 - v_s^2}$, so

$$v_s^2 x_0^2 (1 - v_s^2)^{-1} > d^2 + (1 - v_s^2)^{-1} v_s^2 t^2. \tag{2.28}$$

Giving

$$v_s^2 > \frac{d^2}{x_0^2 - t^2 + d^2}. \tag{2.29}$$

So in order to get a CTC via this route, with $v_s < 1$ we require $x_0^2 - t^2 > 0$.

To summarise: using the cosmic strings we can form "shorts cuts" to allow a rocket travelling at sub-luminal velocities to join two spacelike separated events in the surface $y = 0$. As these events are spacelike separated there exists a Lorentz frame in which they are simultaneous. Making these two events simultaneous can be achieved by moving the two strings in opposite directions. We can now traverse both strings, starting at event A and finally returning to event A — when and where we started.

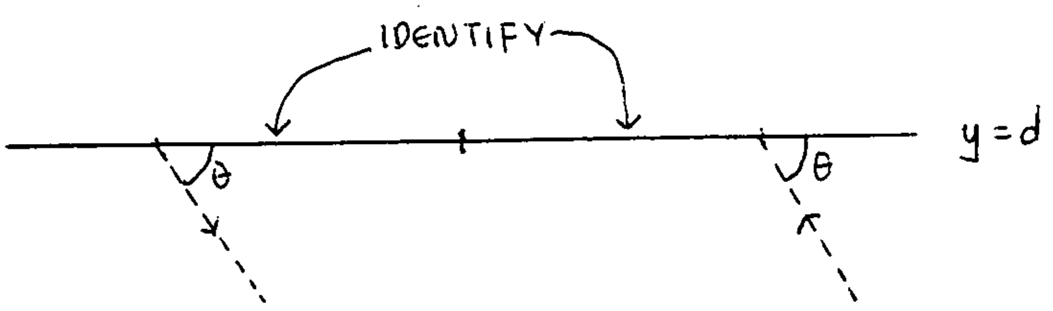

Figure 2.6: *A null geodesic being scattered by a cosmic string.*

2.3.1 The Global Structure of Gott Space

In this section we examine the global structure of Gott space, show that it consists of chronal and non-chronal regions and investigate the nature of the chronology horizon.

As a starting point we will first examine the behaviour of null geodesics in the spacetime. Consider first a null geodesic incident on a static cosmic string (with π deficit angle).

Imagine a null geodesic hitting the string's wedge at $(T, X, y = d)$, with angle ϕ. The geodesic will then appear from the other wedge at $(T, -X, y = d)$ with angle $\phi + \pi$ (see figure (2.6)) . The effect of a cosmic string is to rotate null geodesics by an amount equal to the deficit angle of the string. In the Gott space the strings are moving; we have to see what result this has on the behaviour of the null geodesics. Before we continue, define the following three coordinate systems:

- (t_L, x_L, y_L) - centre of mass/laboratory frame coordinates

- (t_1, x_1, y_1) - coordinates in the rest frame of string 1
- (t_2, x_2, y_2) - coordinates in the rest frame of string 2

These are related as follows

$$t_L = t_1 \cosh a + x_1 \sinh a = t_2 \cosh a - x_2 \sinh a, \tag{2.30}$$

$$x_L = x_1 \cosh a + t_1 \sinh a = x_2 \cosh a - t_2 \sinh a, \tag{2.31}$$

$$y_L = y_1 = y_2, \tag{2.32}$$

$$z_L = z_1 = z_2. \tag{2.33}$$

In the rest frame of string 1 the following points are identified:

$$(t_1, x_1, y_1 = d) \equiv (t_1, -x_1, y_1 = -d), \tag{2.34}$$

and in the rest frame of string 2 the following points are identified

$$(t_2, x_2, y_2 = -d) \equiv (t_2, -x_2, y_2 = -d). \tag{2.35}$$

What do these identifications look like in the laboratory frame? To see what happens in the lab frame, we boost to the rest frame of the string, rotate by an angle of π, and then boost back to the lab frame.

So for string 1

$$\begin{pmatrix} t_L \\ x_L \\ y_L \end{pmatrix} \to \begin{pmatrix} \cosh a & \sinh a & 0 \\ \sinh a & \cosh a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh a & -\sinh a & 0 \\ -\sinh a & \cosh a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_L \\ x_L \\ y_L \end{pmatrix}$$

So in the lab frame

$$\begin{pmatrix} t_L \\ x_L \\ y_L = d \end{pmatrix} \tag{2.36}$$

is identified with

$$\begin{pmatrix} \cosh 2a & -\sinh 2a & 0 \\ \sinh 2a & -\cosh 2a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_L \\ x_L \\ y_L = d \end{pmatrix} \tag{2.37}$$

And for string 2 in a similar fashion

$$\begin{pmatrix} t_L \\ x_L \\ y_L = -d \end{pmatrix} \tag{2.38}$$

is identified with

$$\begin{pmatrix} \cosh 2a & \sinh 2a & 0 \\ -\sinh 2a & -\sinh 2a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_L \\ x_L \\ y_L = -d \end{pmatrix} \tag{2.39}$$

Now consider the behaviour of null geodesics spiralling round the Gott spacetime (figure (2.8)). Define the following angles:

$$\cos\phi_1 = -\frac{dx_1}{dt_1}\ ,\ \sin\phi_1 = \frac{dy_1}{dt_1}, \tag{2.40}$$

$$\cos\phi_L = -\frac{dx_L}{dt_L}\ ,\ \sin\phi_L = \frac{dy_L}{dt_L}, \tag{2.41}$$

$$\cos\phi_2 = -\frac{dx_2}{dt_2}\ ,\ \sin\phi_2 = \frac{dy_2}{dt_2}. \tag{2.42}$$

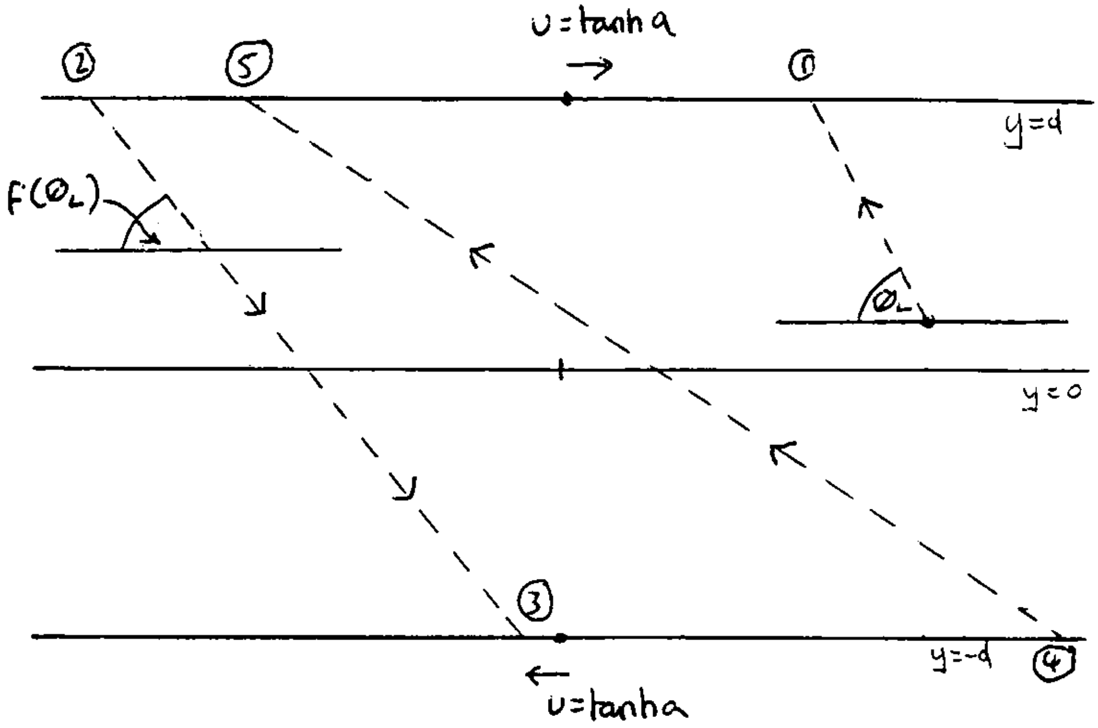

Figure 2.7: *A null geodesic spiralling round both strings. Follow the null ray around the spacetime in numerical order.*

Consider a geodesic incident on string 1 with angle ϕ_L (as measured in the lab frame) where $0 < \phi_L < \pi$.

$$\begin{aligned}\cot\phi_L &= -\frac{dx_L}{dy_L} \\ &= -\cosh a\frac{dx_1}{dy_1} - \sinh a\frac{dt_1}{dy_1} \\ &= \cosh a\cot\phi_1 - \sinh a\csc\phi_1.\end{aligned}$$

On crossing string 1 $\phi_1 \to \phi_1 + \pi$. Therefore

$$\cot\phi_L \quad\rightarrow\quad \cosh a\cot\phi_1 + \sinh a\csc\phi_1$$

$$
\begin{aligned}
&= -\cosh a \frac{dx_1}{dy_1} + \sinh a \frac{dt_1}{dx_1} \\
&= \cosh a(-\frac{d}{dy_L}(x_L \cosh a - t_L \sinh a)) \\
&\quad + \sinh a(\frac{d}{dy_L}(t_L \cosh a - x_L \sinh a)) \\
&= -\cosh 2a \frac{dx_L}{dy_L} + \sinh 2a \frac{dt_L}{dy_L} \\
&= \cosh 2a \cot \phi_L + \sinh 2a \csc \phi_L .
\end{aligned}
$$

So on crossing string 1

$$\cot \phi_L \rightarrow \cosh 2a \cot \phi_L + \sinh 2a \csc \phi_L .$$

On crossing the string we want ϕ_L to lie in the range $\pi < \phi_L < 2\pi$, (see figure (2.8)). So

$$\phi_L \rightarrow f(\phi_L) + \pi \tag{2.43}$$

where

$$f(\phi_L) = \cot^{-1}(\cosh 2a \cot \phi_L + \sinh 2a \csc \phi_L), \tag{2.44}$$

and $0 < f(\phi_L) < \pi$.

Now consider a geodesic crossing string 2. In an identical manner to the above we get:

$$\cot \phi_L \rightarrow \cosh 2a \cot \phi_L - \sinh 2a \csc \phi_L .$$

So on crossing string 2

$$\phi_L \rightarrow g(\phi_L), \tag{2.45}$$

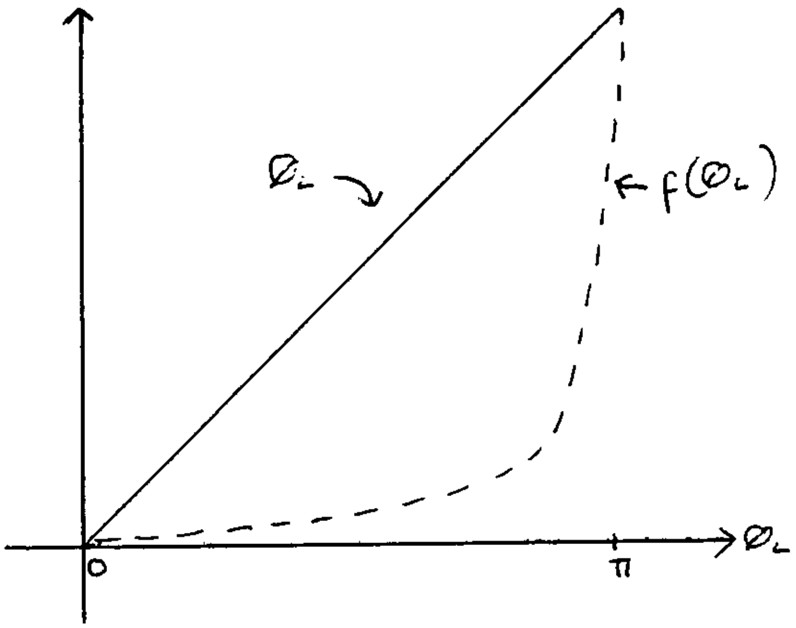

Figure 2.8: *A plot of the function $f(\phi_L)$ against ϕ_L. We can see that ϕ_L is quickly driven to zero after a few trips round the strings.*

where

$$g(\phi_L) = \cot^{-1}(\cosh 2a \cot \phi_L - \sinh 2a \csc \phi_L), \tag{2.46}$$

and $0 < \phi_L < \pi$.

Consider a geodesic spiralling round both strings, initially incident on string 1: Crossing string 1 $\phi_L \to f(\phi_L) + \pi$, crossing string 2 $\phi_L \to g(f(\phi_L) + \pi) = f(f(\phi_L))$.

So after n traversals of the spacetime $\phi_L \to f^{2n}(\phi_L)$. We see from figure (2.8) that ϕ_L rapidly tends to zero with n.

Now consider the geodesic initially incident on string 2: crossing string 2 $\phi_L \to$

$g(\phi_L)$, crossing string 1 $\phi_L \to f(g(\phi_L)) + \pi$. So

$$\phi_L - \pi \to f(g(\phi_L)) = f(f(\phi_L - \pi)). \tag{2.47}$$

In this case $\phi_L - \pi \to 0$ after a number of trips round the strings, hence $\phi_L \to \pi$.

Frequency Shifting of Null Rays

As a null ray crosses a string it will experience a blue (red) shift due to the different times the wedges are identified at in the lab frame. Consider a null geodesic parameterized by affine parameter λ in the Gott spacetime. To calculate the frequency shift of a null ray we will consider the change in t_L with respect to λ as the geodesic passes round both the strings.

Consider a geodesic crossing string 1:

$$\begin{aligned}
\frac{dt_L}{d\lambda} &= \frac{dt_1}{d\lambda}\cosh a + \frac{dx_1}{d\lambda}\sinh a \\
&= \frac{dt_1}{d\lambda}\cosh a + \frac{dx_1}{dt_1}\frac{dt_1}{d\lambda}\sinh a \\
&= \frac{dt_1}{d\lambda}(\cosh a - \sinh a \cos\phi_1).
\end{aligned}$$

On crossing string 1, $\phi_1 \to \phi_1 + \pi$ and $\frac{dt_1}{d\lambda}$ remains unchanged. So

$$\frac{dt_L}{d\lambda} \to \frac{dt_1}{d\lambda}(\cosh a + \sinh a \cos\phi_1).$$

So crossing string 1:

$$\frac{dt_L}{d\lambda} \to \frac{dt_L}{d\lambda}\left(\frac{\cosh a + \sinh a \cos\phi_1}{\cosh a - \sinh a \cos\phi_1}\right). \tag{2.48}$$

Similarly crossing string 2:

$$\frac{dt_L}{d\lambda} \rightarrow \frac{dt_L}{d\lambda}\left(\frac{\cosh a - \sinh a \cos\phi_1}{\cosh\ + \sinh a \cos\phi_1}\right). \tag{2.49}$$

We can re-write (2.48) and (2.49) using the follow identities:

$$\cot\phi_L = \cosh a \cot\phi_1 - \sinh a \csc\phi_1 \tag{2.50}$$

$$\csc\phi_L = \cosh a \csc\phi_1 - \sinh a \cot\phi_1 \tag{2.51}$$

$$\cot\phi_L = \cosh a \cot\phi_2 + \sinh a \csc\phi_2 \tag{2.52}$$

$$\csc\phi_L = \cosh a \csc\phi_2 + \sinh a \cot\phi_2 \tag{2.53}$$

As $\phi_1 \neq 0, \pi$ we can divide the top and bottom of (2.48) by $\sin\phi_1$. So we get:

$$\begin{aligned}\frac{dt_L}{d\lambda} &\rightarrow \frac{dt_L}{d\lambda}\left(\frac{\cosh a \csc\phi_1 + \sinh a \cot\phi_1}{\cosh a \csc\phi_1 - \sinh a \cot\phi_1}\right) \\ &= \frac{dt_L}{d\lambda}\left(\frac{\cosh 2a \csc\phi_L + \sinh 2a \cot\phi_L}{\csc\phi_L}\right).\end{aligned} \tag{2.54}$$

Similarly (2.49) becomes

$$\frac{dt_L}{d\lambda} \rightarrow \frac{dt_L}{d\lambda}\left(\frac{\cosh 2a \csc\phi_L - \sinh 2a \cot\phi_L}{\csc\phi_L}\right). \tag{2.55}$$

Now recall that on crossing string 1

$$\begin{aligned}\cot\phi_L &\rightarrow \cot\phi'_L = \cosh 2a \cot\phi_L + \sinh 2a \csc\phi_L, \\ \csc\phi_L &\rightarrow \csc\phi'_L = -\cosh 2a \csc\phi_L - \sinh 2a \cot\phi_L,\end{aligned}$$

and crossing string 2

$$\begin{aligned}\cot\phi_L &\rightarrow \cot\phi''_L = \cosh 2a \cot\phi_L - \sinh 2a \csc\phi_L, \\ \csc\phi_L &\rightarrow \csc\phi''_L = -\cosh 2a \csc\phi_L + \sinh 2a \cot\phi_L.\end{aligned}$$

We can now look at the frequency shift on going round both strings. Suppose our geodesic is initially incident on string 1, crossing string 1 followed by string 2:

$$\frac{dt_L}{d\lambda} \to \left(\frac{(\cosh 2a \csc \phi'_L - \sinh 2a \cot \phi'_L)}{\csc \phi'_L}\right) \left(\frac{(\cosh 2a \csc \phi_L + \sinh 2a \cot \phi_L)}{\csc \phi_L}\right) \frac{dt_L}{d\lambda}.$$

So we get:

$$\frac{dt_L}{d\lambda} \to (\cosh 4a + \sinh 4a \cos \phi_L) \frac{dt_L}{d\lambda}. \tag{2.56}$$

After a number of trips round both strings $\phi_L \approx 0$ and so

$$\frac{dt_L}{d\lambda} \to A \frac{dt_L}{d\lambda}, \tag{2.57}$$

where $A \approx e^{4a} > 1$. This corresponds to a blue shift.

We can do a similar calculation for null rays initially incident on string 2. Here

$$\frac{dt_L}{d\lambda} \to (\cosh 4a - \sinh 4a \cos \phi_L) \frac{dt_L}{d\lambda}. \tag{2.58}$$

Now after many loops round both strings $\phi_L \to \pi$ and so again

$$\frac{dt_L}{d\lambda} \to A \frac{dt_L}{d\lambda} \tag{2.59}$$

and again we have a blue shift.

So we see that any null geodesic spiralling round and round gets blue shifted. It gets blue shifted more and more as we approach the fixed point angle, 0 and π, on each successive trip.

The Covering Space for Gott Space

In order to consider the Cauchy problem in Gott space we will need to consider the structure of the past light cone of a point in the spacetime. To look at the past light cone of a point we simply trace back all null geodesics from that point, and examine their behaviour as they wrap around the strings. Now as we have seen Gott space is compact in the y-direction ($-d \leq y \leq d$), however, an examination of the past light cones in Gott space will show us that we can form a covering space for Gott space that is defined for all y. This covering space will prove very useful in examining certain properties of the spacetime and when looking at the Cauchy problem.

Before we start looking at the covering space for Gott space we consider the simpler case of that for a single cosmic string spacetime $(M, \mathbf{g})$ with deficit angle π.

Consider a cosmic string located at $(t, x = 0, y = 0)$ with deficit angle π. Let the string wedge lie across the x-axis. The π-wedge cosmic string spacetime is then the $y \leq 0$ region of Minkowski spacetime $(M, \mathbf{g}')$ with the point $(t, x, y = 0)$ identified with the point $(t, -x, y = 0)$, see figure (2.9). Now as already stated a null geodesic incident on a cosmic string of deficit angle π is rotated by an angle π, see figure (2.9). One can form a covering space for the cosmic string spacetime by extending it into the $y > 0$ region of Minkowski space. So for a geodesic in the covering space to represent a geodesic in the physical spacetime we must have points identified by a rotation of angle π in the origin. So such a rotation R_π maps a point (t, x, y) to a point $(t, -x, -y)$. So the π-wedge cosmic string spacetime is then simply the quotient

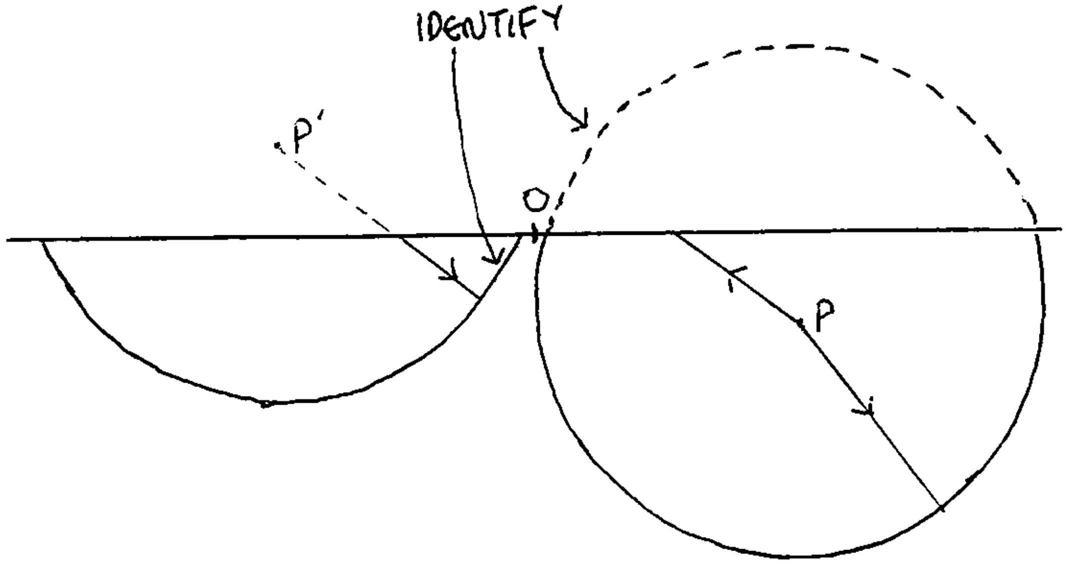

Figure 2.9: *The π-wedge cosmic string spacetime . The points p and p' are identified. A null geodesic is rotated by angle π on traversing the string.*

$(M, \mathbf{g}')/R_\pi$.

Now let us look at the covering space for Gott space. Again the covering space for Gott space $(M_g, \mathbf{g})$ will be Minkowski space $(M, \mathbf{g}')$. To form the covering space we start by looking at the behaviour of null geodesics in the laboratory frame of the physical spacetime. By demanding that the geodesics be straight lines in the covering space we will find the group of isometries on the covering space. Consider a geodesic propagating back in time from the point $(t_L = T, x_L = X, y_L = Y)$ and wrapping around the strings. First we look at geodesics initially incident on string 1 with angle ϕ_L, where $0 < \phi_L < \pi$. The two cosmic strings are placed at $y = d$ and $y = -d$ (see

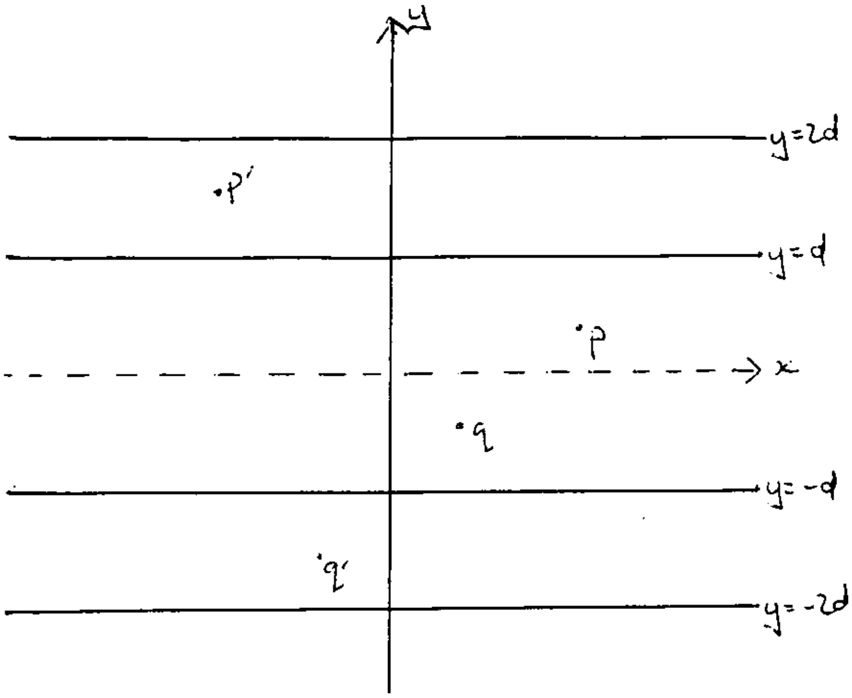

Figure 2.10: *The covering space for the π-wedge cosmic string spacetime. The points p and p' are identified, and the points q and q' are identified.*

figure (2.11)).

The geodesic will hit the top string at

$$t_L = T - (d - Y)\csc\phi_L,\ \ x_L = X - (d - Y)\cot\phi_L. \tag{2.60}$$

On crossing string 1

$$\begin{pmatrix} t_L \\ x_L \end{pmatrix} \rightarrow \begin{pmatrix} \cosh 2a & -\sinh 2a \\ \sinh 2a & -\cosh 2a \end{pmatrix} \begin{pmatrix} T - (d - Y)\csc\phi_L \\ X - (d - Y)\cot\phi_L \end{pmatrix}. \tag{2.61}$$

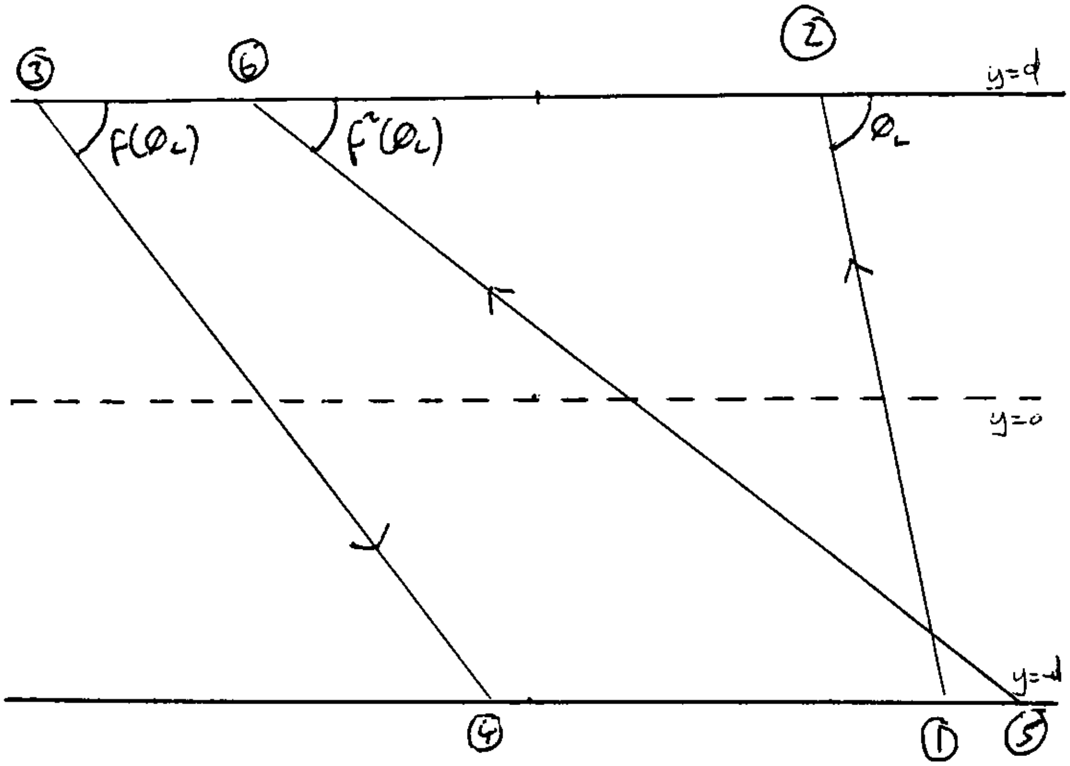

Figure 2.11: *A null geodesic propagating round the Gott spacetime. Follow the numbers.*

The geodesic will hit string 2 at

$$\begin{pmatrix} t_L \\ x_L \end{pmatrix} = \begin{pmatrix} \cosh 2a & -\sinh 2a \\ \sinh 2a & -\cosh 2a \end{pmatrix} \begin{pmatrix} T-(d-Y)\csc\phi_L \\ X-(d-Y)\cot\phi_L \end{pmatrix} + 2d \begin{pmatrix} -\csc f^{-1}(\phi_L) \\ \cot f^{-1}(\phi_L) \end{pmatrix}. \tag{2.62}$$

(The reason we have an $f^{-1}(\phi_L)$ in the above expression is because we are tracing our geodesic backwards, hence $\phi_L \to f^{-1}(\phi_L)$ on crossing string 1.)

Crossing string 2 $\begin{pmatrix} t_L \\ x_L \end{pmatrix}$ becomes

$$\begin{pmatrix} \cosh 2a & \sinh 2a \\ -\sinh 2a & -\cosh 2a \end{pmatrix} \begin{pmatrix} \cosh 2a & -\sinh 2a \\ \sinh 2a & -\cosh 2a \end{pmatrix} \begin{pmatrix} T-(d-Y)\csc\phi_L \\ X-(d-Y)\cot\phi_L \end{pmatrix}$$
$$+2d \begin{pmatrix} \cosh 2a & \sinh 2a \\ -\sinh 2a & -\cosh 2a \end{pmatrix} \begin{pmatrix} -\csc f^{-1}(\phi_L) \\ \cot f^{-1}(\phi_L) \end{pmatrix}. \tag{2.63}$$

We have now completed a complete journey round the spacetime (both strings). Now there is some simplification that can be done.

Recall that

$$\begin{aligned} \cot f(\phi_L) &= \cosh 2a \cot\phi_L + \sinh 2a \csc\phi_L, \\ \csc f(\phi_L) &= \cosh 2a \csc\phi_L + \sinh 2a \cot\phi_L. \end{aligned}$$

Hence

$$\cot\phi_L = \cosh 2a \cot f(\phi_L) - \sinh 2a \csc f(\phi_L), \tag{2.64}$$

$$\csc\phi_L = \cosh 2a \csc f(\phi_L) - \sinh 2a \cot f(\phi_L). \tag{2.65}$$

Using (2.64) and (2.65) we see

$$\cot f(\phi_L) = \cosh 2a \cot f^2(\phi_L) - \sinh 2a \csc f^2(\phi_L), \tag{2.66}$$

$$\csc f(\phi_L) = \cosh 2a \csc f^2(\phi_L) - \sinh 2a \cot f^2(\phi_L). \tag{2.67}$$

Combining (2.66) and (2.67) with (2.64) and (2.65) we get

$$\cot\phi_L = \cosh 4a \cot f^2(\phi_L) - \sinh 4a \csc f^2(\phi_L), \tag{2.68}$$

$$\csc\phi_L = \cosh 4a \csc f^2(\phi_L) - \sinh 4a \cot f^2(\phi_L). \qquad (2.69)$$

We can now repeat this procedure, putting $\cot f^2(\phi_L)$ in terms of $f^3(\phi_L)$ etc...

$$\csc\phi_L = \cosh 2na \csc f^n(\phi_L) - \sinh 2na \cot f^n(\phi_L), \qquad (2.70)$$

$$\cot\phi_L = \cosh 2na \cot f^n(\phi_L) - \sinh 2na \csc f^n(\phi_L), \qquad (2.71)$$

$$\csc f^n(\phi_L) = \cosh 2na \csc\phi_L + \sinh 2na \cot\phi_L, \qquad (2.72)$$

$$\cot f^n(\phi_L) = \cosh 2na \cot\phi_L + \sinh 2na \csc\phi_L. \qquad (2.73)$$

Now go back to our previous expression for $\begin{pmatrix} t_L \\ X_L \end{pmatrix}$:

$$\begin{aligned}
\begin{pmatrix} t_L \\ x_L \end{pmatrix} &= \begin{pmatrix} \cosh 4a & -\sinh 4a \\ -\sinh 4a & \cosh 4a \end{pmatrix} \begin{pmatrix} T-(d-Y)\csc\phi_L \\ X-(d-Y)\cot\phi_L \end{pmatrix} \\
&+ 2d \begin{pmatrix} \cosh 2a & \sinh 2a \\ -\sinh 2a & -\cosh 2a \end{pmatrix} \begin{pmatrix} -\cosh 2a & \sinh 2a \\ -\sinh 2a & \cosh 2a \end{pmatrix} \begin{pmatrix} \csc\phi_L \\ \cot\phi_L \end{pmatrix} \\
&= \begin{pmatrix} \cosh 4a & -\sinh 4a \\ -\sinh 4a & \cosh 4a \end{pmatrix} \begin{pmatrix} T-(d-Y)\csc\phi_L - 2d\csc\phi_L \\ X-(d-Y)\cot\phi_L - 2d\cot\phi_L \end{pmatrix}
\end{aligned}$$

Our geodesic will return to string 1 at

$$\begin{aligned}
\begin{pmatrix} t_L \\ x_L \end{pmatrix} &= \begin{pmatrix} \cosh 4a & -\sinh 4a \\ -\sinh 4a & \cosh 4a \end{pmatrix} \begin{pmatrix} T-(d-Y)\csc\phi_L - 2d\csc\phi_L \\ X-(d-Y)\cot\phi_L - 2d\cot\phi_L \end{pmatrix} \\
&- 2d \begin{pmatrix} \csc f^{-2}(\phi_L) \\ \cot f^{-2}(\phi_L) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \cosh 4a & -\sinh 4a \\ -\sinh 4a & \cosh 4a \end{pmatrix} \begin{pmatrix} T - (3d - Y)\csc\phi_L - 2d\csc\phi_L \\ X - (3d - Y)\cot\phi_L - 2d\cot\phi_L \end{pmatrix}$$

We can now take another trip round both strings. Repeat the above procedure but now replace $T - (d - Y)\csc\phi_L$ and $X - (d - Y)\cot\phi_L$ in the above expression for t_L and x_L, and replace ϕ by $f^{-2}(\phi)$. So after two trips round the spacetime the geodesic will hit string one at

$$\begin{pmatrix} \cosh 4a & -\sinh 4a \\ -\sinh 4a & \cosh 4a \end{pmatrix}^2 \begin{pmatrix} T - (d - Y)\csc\phi_L - 6d\csc\phi_L \\ X - (d - Y)\cot\phi_L - 6d\cot\phi_L \end{pmatrix}$$

After n trips round both strings our geodesic will return to string 1 at

$$\begin{pmatrix} t_L \\ x_L \end{pmatrix} = \begin{pmatrix} \cosh 4na & -\sinh 4na \\ -\sinh 4na & \cosh 4na \end{pmatrix} \begin{pmatrix} T - [(4n-1)d - Y]\csc\phi_L \\ X - [(4n-1)d - Y]\cot\phi_L \end{pmatrix} \tag{2.74}$$

So after exactly n trips round the spacetime the general position of our geodesic will be:

$$\begin{aligned} \begin{pmatrix} t_L \\ x_L \end{pmatrix} &= \begin{pmatrix} \cosh 4na & -\sinh 4na \\ -\sinh 4na & \cosh 4na \end{pmatrix} \begin{pmatrix} T - [(4n-1)d - Y]\csc\phi_L \\ X - [(4n-1)d - Y]\cot\phi_L \end{pmatrix} \\ &\quad - \begin{pmatrix} r \\ r\cos f^{-2n}(\phi_L) \end{pmatrix} \\ y_L &= -d + r\sin f^{-2n}(\phi_L) \end{aligned} \tag{2.75}$$

In equation (2.75) r is an affine parameter along the geodesic. f^{-2n} will be the angle with which the geodesic emerges from string 1.

(2.75) can be re-written as

$$\begin{aligned}
\begin{pmatrix} t_L \\ x_L \end{pmatrix} &= \begin{pmatrix} \cosh 4na & -\sinh 4na \\ -\sinh 4na & \cosh 4na \end{pmatrix} \begin{pmatrix} T - [(4n-1)d - Y]\csc\phi_L \\ X - [(4n-d)d - Y]\cot\phi_L \end{pmatrix} \\
& \quad - \sin f^{-2n}(\phi_L) \begin{pmatrix} r\csc f^{-2n}(\phi_L) \\ -\cot f^{-2n}(\phi_L) \end{pmatrix} \\
y_L &= -d + r\sin f^{-2n}
\end{aligned}$$

Giving

$$\begin{aligned}
\begin{pmatrix} t_L \\ x_L \end{pmatrix} &= \begin{pmatrix} \cosh 4na & -\sinh 4na \\ -\sinh 4na & \cosh 4na \end{pmatrix} \begin{pmatrix} T - R\csc\phi_L \\ X - R\cot\phi_L \end{pmatrix} && (2.76) \\
&= \begin{pmatrix} \cosh 4na & -\sinh 4na \\ -\sinh 4na & \cosh 4na \end{pmatrix} \begin{pmatrix} T \\ X \end{pmatrix} - \begin{pmatrix} R\csc f^{-2n}(\phi_L) \\ R\cot f^{-2n}(\phi_L) \end{pmatrix} \\
y_L &= -d - ((4n-1)d - Y - R) \\
&= -4nd + Y + R && (2.77)
\end{aligned}$$

where

$$R = (4n-1)d - Y + r\sin f^{-2n}(\phi_L). \qquad (2.78)$$

In a similar fashion we can deduce then general position of a geodesic after $n + \frac{1}{2}$ trips round both strings:

$$\begin{aligned}
\begin{pmatrix} t_L \\ x_L \end{pmatrix} &= \begin{pmatrix} \cosh(4n+2)a & -\sinh(4n+2)a \\ \sinh(4n+2)a & -\cosh(4n+2)a \end{pmatrix} \begin{pmatrix} T - R'\csc\phi_L \\ X - R'\cot\phi_L \end{pmatrix} && (2.79) \\
&= \begin{pmatrix} \cosh(4n+2)a & -\sinh(4n+2)a \\ \sinh(4n+2)a & -\cosh(4n+2)a \end{pmatrix} \begin{pmatrix} T \\ X \end{pmatrix} - \begin{pmatrix} R'\csc f^{-2n-1}(\phi_L) \\ R'\cot f^{-2n-1}(\phi_L) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned} y_L &= d - Y + (4n+1)d + R' \\ &= -Y + (4n+2)d + R'. \end{aligned} \tag{2.80}$$

where

$$R' = (4n+1)d - Y + r\sin f^{-2n-1}(\phi_L). \tag{2.81}$$

So in order that the geodesic

$$t_L = T - R\csc\phi_L, \tag{2.82}$$

$$x_L = X - R\cot\phi_L, \tag{2.83}$$

$$y_L = Y + R, \tag{2.84}$$

in the covering space represent a geodesic in the physical space that passes n times round both the strings we must identify the point (t, x, y) with the point $C^n(t, x, y)$, where C is the generator of a subgroup G_1 of the Lorentz group which maps the point (t, x, y) to the point

$$(t\cosh 4a - x\sinh 4a, x\cos 4a - t\sinh 4a, y - 4d). \tag{2.85}$$

So we identify points of the form:

$$(t\cosh 4na - x\sinh 4na, x\cos 4na - t\sinh 4na, y - 4nd). \tag{2.86}$$

Similarly in order that a geodesic that passes $n + \frac{1}{2}$ times round both strings be a geodesic in the covering space we must identify points under the action of a group G_2, where a generator D of G_2 maps the point (t, x, y) to the point

$$(t\cosh 2a - x\sinh 2a, -x\cosh 2a + t\sinh 2a, -y + 2d). \tag{2.87}$$

So we identify points of the form:

$$(t\cosh(4n+2)a - x\sinh(4n+2)a, -x\cosh(4n+2)a + t\sinh(4n+2)a, -y + (4n+2)d). \tag{2.88}$$

identified for all n.

So Gott space $(M_g, \mathbf{g})$ is the quotient $(M, \mathbf{g}')/G$, where G is generated by C and D. We can see that the covering space for Gott space with cosmic string of deficit angle π is in fact the same as that for Grant space, but with the addition isometries given by G_2.

Now one cannot complete a closed timelike curve unless one traverses both the strings in Gott space. This then means that all of the interesting causal behaviour of Gott space is due to the isometries G_1. Therefore the non-causal behaviour of Gott space will be identical to that of Grant space.

Now that we have established a covering space for Gott space we can use it to prove various things about the causal structure of Gott space, and in fact Grant space.

The Causal Structure of Grant Space and Gott Space

Using the covering space we will see that there is a region that contains no CTCs, and therefore there exists a chronology horizon.

In order for a point (t, x, y) to lie on a CTC it must be such that you start at (t, x, y), spiral round the strings n times (always moving along a timelike curve) and finally return to (t, x, y). In the covering space this is equivalent to joining (t, x, y) to its image point $(t \cosh 4na - x \sinh 4na, x \cosh 4na - t \sinh 4na, y - 4nd)$ by a timelike curve. For a point to be timelike connected to its image point we must have:

$$(t - t \cosh 4na + x \sinh 4na)^2 - (x - x \cosh 4na + t \sinh 4na)^2 - (y - y + 4nd)^2 > 0. \quad (2.89)$$

This gives:

$$2(t^2 - x^2) - 2t(t \cosh 4na - x \sinh 4na) + 2x(x \cosh 4na - t \sinh 4na) > (4nd)^2.$$

And for closed timelike curves to exist we require:

$$x^2 - t^2 > \frac{8n^2 d^2}{\cosh 4na - 1} > 0. \quad (2.90)$$

So a point (t, x, y) lying in the region $x^2 - t^2 > 0$ can lie on a CTC, but a point in the region $x^2 - t^2 < 0$ cannot. We therefore have a chronology horizon at $x^2 - t^2 = 0$.

So what can be deduce about the chronology horizon? Well it consists of two null planes $t - x = 0$, $t + x = 0$ and is generated by null geodesics that come from past null infinity and extend to future null infinity. (As we will see later this chronology horizon is not "compactly generated" as is the case in the wormhole and Misner space examples).

It has be postulated by Thorne that the chronology horizon is the $n \to \infty$ limit of the n^{th} polarised hypersurface. A polarised hypersurface is generated by self-intersecting null geodesics. A self-intersecting null geodesic is a null geodesic that spirals n time round both strings eventually returning to its point of origin, but in

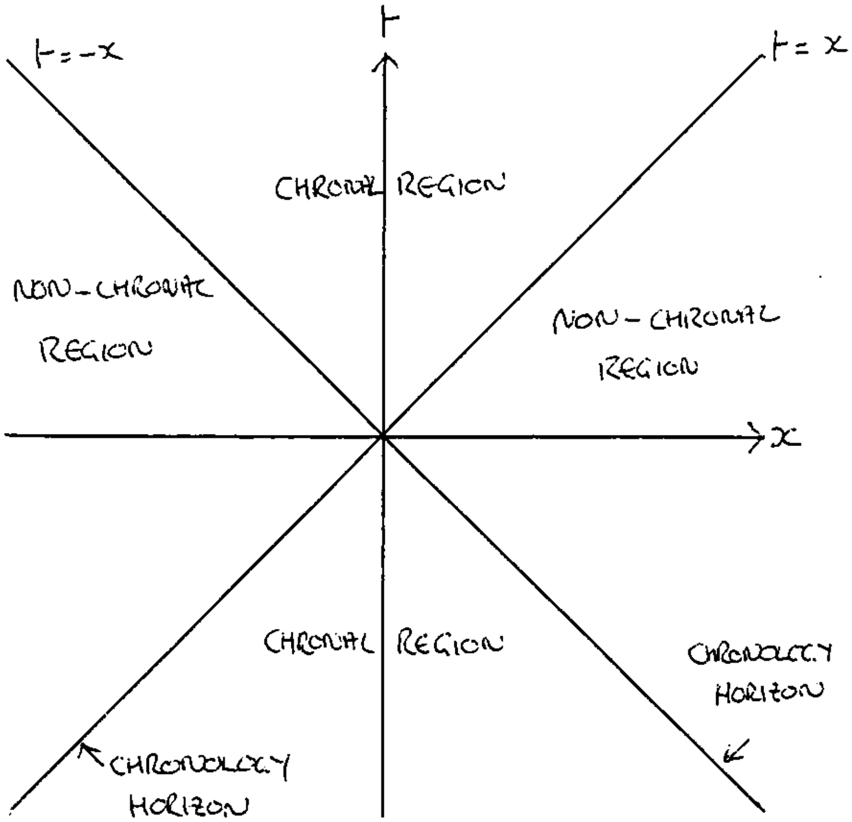

Figure 2.12: *The causal structure of Gott space.*

contrast to closed null geodesics it need not have the same tangent vector on its return. To get an expression for points that lie on these self-intersecting null geodesics we consider tracing a null geodesic from a point (t, x, y), round both strings and back to the point (t, x, y). This can be done easily in the covering space by consider a point joined to its image point by a null curve. To do this we replace the > 0 in (2.90) by $= 0$. So we see that the n^{th} polarised hypersurface is generated by geodesics with

$$x^2 - t^2 = \frac{8n^2d^2}{\cosh 4na - 1}. \tag{2.91}$$

Taking the $n \to \infty$ limit of (2.91) does indeed yield $t^2 - x^2 = 0$ — the chronology

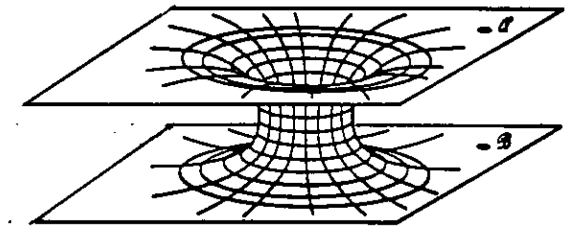

Figure 2.13: *A wormhole joining two asymptotically flat universes*

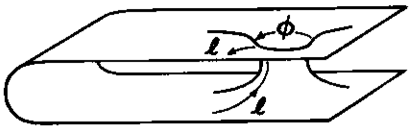

Figure 2.14: *A wormhole joining two asymptotically flat regions of one universe*

horizon.

2.4 Time Machines from Wormholes

We will now look at time machines constructed from wormholes, and in particular the so called "twin paradox" spacetime [MTY,FMN].

First we need to define exactly what is meant by a wormhole. Wormholes are again hypothetical objects in general relativity and seem to originate in Wheeler's ideas on quantum gravity.

The wormhole spacetimes we study here are four dimensional spacetimes that are empty and flat (globally Minkowskian), except for a single wormhole. Obviously we would like to be able to traverse our wormholes (if we are to use them for time travel), and in order to do so the wormholes must be threaded by fields that violate the weak energy condition [MTY]. Such fields may exist in quantum theory, one example being the Casimir effect. An example of a wormhole is provided by the "throat" of the Kruskal solution (see figure (2.13)). This wormhole may be modified to join two apparently distant regions in the same universe (see figure (2.14). However the wormholes that we will use to construct a time-machine will not be of this type. Rather they will be obtained my making various identifications in Minkowski space. In the simplest example of a wormhole spacetime the mouths of the wormhole are at rest and the throat is infinitesimally short. Consider two worldlines γ_1 and γ_2, the first located at $x = 0$, $y = 0$ and $z = 0$, the second at $x = a$, $y = 0$ and $z = 0$. Let τ measure proper time along the world-lines. Through each value τ on γ_1 and γ_2 there passes a unique flat spacelike hypersurface $S_1(\tau)$ and $S_2(\tau_2)$. On each of these surfaces we remove a three-ball of radius b centred on the world-lines, which we denote $B_1(\tau)$ and $B_2(\tau)$. We now identify the point p_1 on $B_1(\tau)$ with the point p_2 on $B_2(\tau)$ obtained by reflecting in $x = 0$ and then translating (see figure (2.15)). We also change the orientation of the normal so that something entering $B_1(\tau)$ is leaving $B_2(\tau)$ and vice-versa. Since the intrinsic three-geometries of the world-tubes $B_1 \equiv \{B_1(\tau), \ -\infty < \tau < \infty\}$ and $B_2 \equiv \{B_2(\tau), -\infty < \tau < \infty\}$ agree such an identification makes sense. However because of the change in direction of the normal there is a mismatch of the extrinsic curvature of magnitude $\frac{2}{b}$, which results in a $\delta-$function curvature of magnitude $O(\frac{1}{b})$ at the mouths of the wormhole.

In order to convert our wormhole into a time machine it will be necessary to move one mouth of the wormhole. It was shown in [MTY,FMN] that as long as the acceleration of the wormhole mouth is small compared to $\frac{1}{b}$, then there is no

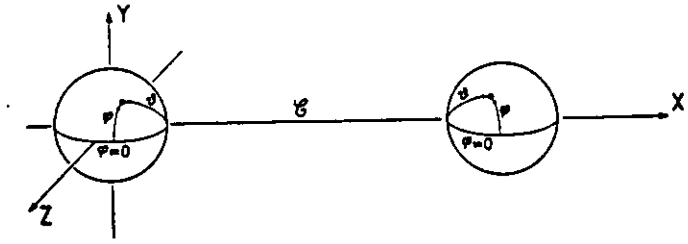

Figure 2.15: *A timelike slice of the wormhole spacetime. Points with the same value of θ and ϕ are identified.*

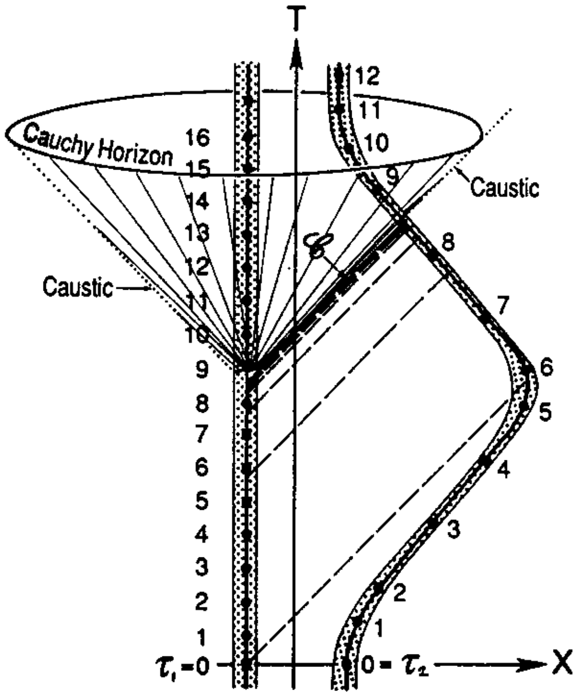

Figure 2.16: *The conversion of a wormhole into a time-machine. This figure shows the world line of the two wormhole mouths. Proper time, as measured by the wormhole throat, is indicated on the world lines. Also notice the chronology horizon.*

significant change to the wormholes internal structure. In this example the closest the mouths ever come to each other (a distance a as measured in the centre of mass frame) is large compared to the wormhole radius b. As a result the wormhole radius would appear so small that all one would see is the world-lines along which they move.

To obtain our CTCs it is necessary for us to perform a "twin paradox" type trip on one of the mouths. The two mouths are initially at rest with separation $a \gg b$ and with proper times $\tau_1 = \tau_2 = 0$ at both mouths. Keep mouth one fixed (inertial) and move mouth two outwards with high speed (but low acceleration) and then bring it back to its starting point (see figure (2.16)). In such a spacetime CTCs form.

We may still construct B_1 and B_2 as before, but due to the acceleration the intrinsic geometries of the world-tubes no longer agree. However provided the acceleration is small compared with $\frac{1}{b}$ only a small perturbation of the flat metric in a neighbourhood of the mouths is needed to make the intrinsic geometries compatible. We can therefore identify points on the two world-tubes as before to obtain a wormhole. There will again be a $\delta-$function curvature of order $\frac{1}{b}$ at the mouths of the wormhole.

Due to special relativistic effects mouth two will have aged less in the "external spacetime", but as seen through the wormhole mouth no such aging will have occurred (as proper times as identified through the wormhole mouth). Provided the time taken to travel between mouths one and two is less than the proper-time difference on each mouth then CTCs can be formed. Let us illustrate this (see figure (2.16)) : we start at mouth one at external time $T = 11$ (which is the same as the proper-time on mouth one), travel along a timelike trajectory to mouth two. As the mouths are identified under proper-time we will enter mouth two at $\tau_2 = 9$ and emerge from mouth one at

$\tau_1 = T = 9$.

As stated above the wormholes are not threaded by CTCs until the return journey, and we have a chronology horizon $\mathcal{H}$ separating the causal and causality violating regions. We will now take a more detailed look at the structure of $\mathcal{H}$ in this example.

$\mathcal{H}$ has several important geometrical properties:

1. CTCs occur to the future of the CNG going through $(8.5, 0, 0, 0)$. The null cone of this point is our chronology horizon. On the other hand if we take S to be the surface $T = 0$ then $H^+(S)$ is $\mathcal{H}$. So then $\mathcal{H}$ is a Cauchy horizon of an edge less achronal set and by proposition 6.5.3 of [HE] is generated by null geodesics with no past end point.

2. No two events on the chronology horizon can be joined by a timelike curve. Now $\mathcal{H}$ is a null surface, and so a future directed timelike curve can only cross it in one direction — from the region without CTCs to the region with CTCs. One may think that due to the CTCs beyond $\mathcal{H}$ it may be possible to join two events (A and B) on $\mathcal{H}$ by a future directed timelike curve k — this is not so. The line k crossed $\mathcal{H}$ at event A and enters the region with CTCs. As $\mathcal{H}$ is null k cannot leave this region and so it must arrive at B from the side of $\mathcal{H}$ containing CTCs. If we extend k slightly beyond B will travel from the region with the CTCs to the region without — which is impossible for k travelling in a locally future direction.

3. No two generators of the chronology horizon can cross each other when

followed into the past, and a generator cannot cross itself. (Crossing of generators would imply that generators could contain point joinable by timelike curves — violating (2)).

Using these properties we can deduce the properties of $\mathcal{H}$. The spacetime is axisymmetric about the line of centres between the wormhole mouths, so that $\frac{\partial}{\partial\phi}$ is a Killing vector (where ϕ is shown in figure (2.15)). As this implies the spacetime looks the same whatever the value of ϕ then the chronology horizon is also axisymmetric. Generators of $\mathcal{H}$ will lie along constant ϕ, and without loss of generality we may consider the case $\phi = 0$. Details of the horizon's structure are somewhat obscured by taking the wormhole's throat to be infinitesimally thin. We therefore now consider a throat with finite, but very small thickness.

It appears that the only way for null geodesics to satisfy (1), (2) and (3) is if they asymptote to one or more closed null geodesics (CNGs) when followed in the past.

So where do these closed null geodesics occur? Consider figure (2.17) with three possible asymptotes. The CNG $\mathcal{E}$ can not be an asymptote as it crosses itself. $\mathcal{C}$ and $\mathcal{D}$ cannot both be asymptotes as they cross each other. Now the above curve $\mathcal{C}$ has shorter spiral length than $\mathcal{D}$. It will thus become a CNG before $\mathcal{D}$ does. Therefore $\mathcal{C}$ lies to the past of the CNG $\mathcal{D}$ and so is a candidate for the asymptote. Extending this argument for other CNGs reveals that $\mathcal{C}$ is indeed the asymptote we seek.

To summarise: In this example the chronology horizon is generated by null geodesics that asymptote to the CNG C when followed to the past. Such CNGs

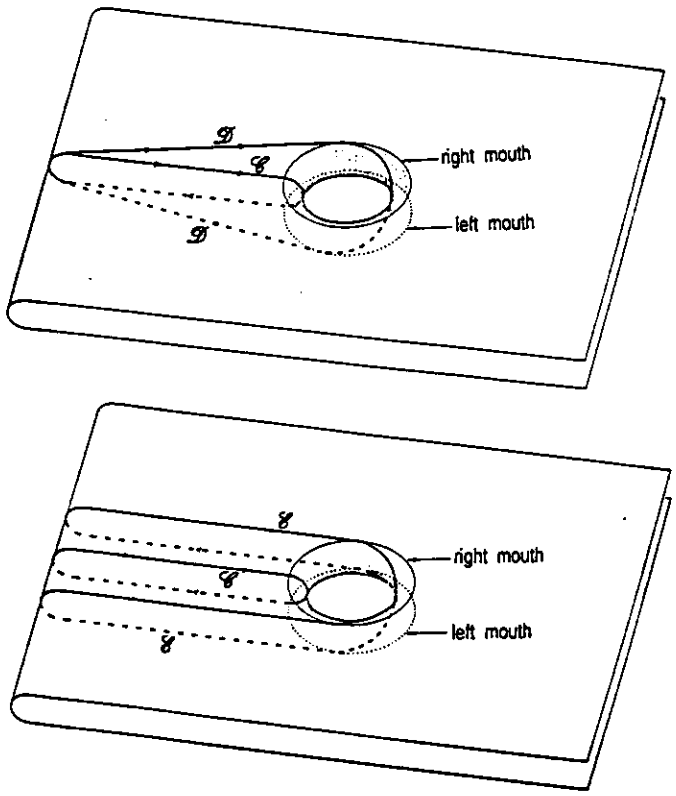

Figure 2.17: *Spatial depiction of the possible asymptote CNGs in the wormhole spacetime.*

are called *fountains* because null geodesics seem to "spring out" of them [Ha]. $\mathcal{H}$ is smooth except for caustics along the x-axis to the right of mouth two and to the left of mouth one. Such horizons are said to be *compactly generated* as all the generators enter and remain in a compact region when followed into the past.

The above behaviour is assumed to be generically true for compactly generated horizons, but there is no hard evidence to prove that this is indeed the case.

Now that we have a detailed picture of $\mathcal{H}$ we are in a position to look at the Cauchy problem in this spacetime. This we will discuss in the next chapter.

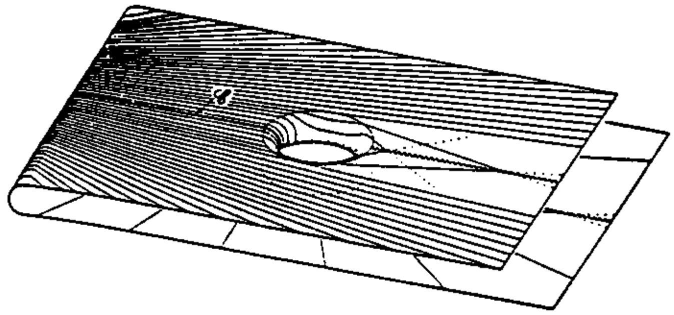

Figure 2.18: *A spatial depiction of the null generators of the chronology horizon. Notice how the generators appear to peel off the CNG C. Also notice the caustic to the right of the wormhole mouth.*

Chapter 3

THE CAUCHY PROBLEM

Physicists have always been reluctant to consider the possibility of constructing time machines. One way to get around the existence of time machines would be to dismiss, perhaps by some physical argument, all objects from which time machines are constructed (such as wormholes and cosmic strings for example). It is not clear that this can be done. Another possibility is to consider the stability of the chronology horizons that appear in many of these CTC spacetimes. This has already been done for the wormhole spacetime [Ly] [KT], Gott/Grant space [Gr] [B] and Misner space [HK]. The approaches in these papers was to consider semi-classical theory, and look at quantum fluctuations of a scalar field on a fixed background. Rather than look at quantum stability of the chronology horizon, the approach in this thesis will be to consider their stability under perturbations due to classical fields. We shall consider the propagation of a mass-less, test scalar field Φ in these spacetimes. Initial data for these fields will be given on some spacelike hypersurface preceding the chronology horizon. There are two possible outcomes for the propagation of the field: firstly, the perturbations to the spacetime build up in such a way that the stress-energy tensor of the scalar field becomes divergent at the horizon, rendering the chronology horizon classical unstable and so (maybe) preventing the causality violating region from evolving. The second possibility is that the stress-energy tensor is finite and the horizon is classically stable to such perturbations. If this is the case we would like to know if a well defined Cauchy problem exists for Φ and furthermore does the wave equation have a unique solution that depends continuously on the initial data. Demanding a well defined Cauchy problem may constrain our initial data, and might prevent certain types of unacceptable behaviour from occurring. One type of behaviour that most physicists find unacceptable is that of going back and changing the past. If we think of this in terms of the Cauchy problem not changing the past is equivalent to demanding that Φ is unique and depends continuously on the initial data, that is, Φ is not altered by data propagating from the future. To ensure that unacceptable behaviour, such as changing that past, does not occur Novikov argued that events on CTCs are already guaranteed to be self-consistent; they influence each other around the closed curve in a self-adjusted, cyclical way.

We will now study the Cauchy problem in the various spacetimes that we have already discussed.

3.1 Misner Space

We first consider the Cauchy problem in two dimensional Misner space. Misner space shares important features with the wormhole spacetimes, and this simple example may give an indication of what happens in the wormhole case.

Recall that Misner space has the following metric

$$ds^2 = 2dX'dT' + T'dX'^2. \quad (3.1)$$

Consider solving the wave equation

$$\Box\Phi = 0, \quad (3.2)$$

with data on some initial surface given by $T' = T'_0 < 0$ (in the globally hyperbolic region).

$$\Phi(T'_0, X') = \Psi_0(X'), \ \frac{\partial\Phi}{\partial T'}(T'_0, X') = V_0(X'). \quad (3.3)$$

Introduce new coordinates u and v that make the metric conformally flat. Putting

$$u = X', \ v = X' + 2\ln T',$$

we get the new metric

$$ds^2 = \exp(\frac{v-u}{2})dudv.$$

The wave equation now takes the form

$$\frac{\partial^2\Phi}{\partial u \partial v} = 0. \quad (3.4)$$

(3.4) has solutions of the form

$$\Phi(u, v) = f(u) + g(v), \quad (3.5)$$

or in terms of the original coordinates

$$\Phi(T', X') = f(X') + g(X' + 2\ln T'). \quad (3.6)$$

This has to satisfy the initial conditions (3.3) at $T' = T'_0$

$$\Psi_0(X') = \Phi(X')|_{T'=T'_0} = f(X') + g(X' + 2\ln T'_0), \tag{3.7}$$

and

$$\begin{aligned} V_0(X') &= \left.\frac{\partial \Phi}{\partial T'}\right|_{T'=T'_0} \\ &= \left(\frac{\partial g}{\partial T'}\right)_{T'=T'_0} \\ &= \left(\frac{2}{T'}\frac{\partial g}{\partial X'}\right)_{T'=T'_0} \\ &= \left.\frac{2}{T'_0}\frac{\partial g}{\partial X'}\right|_{T'=T'_0}. \end{aligned} \tag{3.8}$$

So we get

$$\frac{\partial g}{\partial X'} = \frac{T'_0}{2} V_0(X') \tag{3.9}$$

$$\Rightarrow \quad g(X') = \frac{T'_0}{2}\int^{X'} V_0(w)dw \tag{3.10}$$

$$\Rightarrow \quad g(X' + 2\ln T') = \frac{T'_0}{2}\int^{X'+2\ln T'} V_0(w)dw. \tag{3.11}$$

We get from (3.7)

$$\begin{aligned} f(X') &= \Psi_0(X') - g(X') \\ &= \Psi_0(X') - \frac{T'_0}{2}\int^{X'} V_0(w)dw. \end{aligned} \tag{3.12}$$

And so from (3.6)

$$\begin{aligned} \Phi(T', X') &= \Psi_0(X') - \frac{T'_0}{2}\int^{X'} V_0(w)dw + \frac{T'_0}{2}\int^{X'+2\ln T'} V_0(w)dw \\ &= \Psi_0(X') + \frac{T'_0}{2}\int_{X'}^{X'+2\ln T'} V_0(w)dw. \end{aligned} \tag{3.13}$$

Now consider the stress-energy tensor of our scalar field on the background spacetime.

$$T_{\mu\nu} = \frac{1}{4\pi}(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}), \tag{3.14}$$

and the stress-energy scalar

$$T_{\mu\nu}T^{\mu\nu} = \frac{1}{32\pi^2}(\Phi_{,\mu}\Phi^{,\mu})^2. \tag{3.15}$$

Calculating these in the (T', X') coordinates gives:

$$T_{00} = \frac{1}{4\pi}\frac{T'^2_0}{T'^2}V_0^2(X' + 2\ln T') \tag{3.16}$$

$$T_{01} = T_{10} = \frac{1}{8\pi}\frac{T'^2_0}{T'^2}V_0^2(X' + 2\ln T') \tag{3.17}$$

$$T_{11} = \frac{1}{4\pi}\left[\left(\frac{T'_0}{2}V_0(X' + 2\ln T')\right)^2 + \left(\Psi'_0(X) - \frac{T_0}{2}V_0(X)\right)^2\right], \tag{3.18}$$

and

$$T_{\mu\nu}T^{\mu\nu} = \left[\Psi'_0(X') - \frac{T'_0}{2}V_0(X')\right]\frac{T'^2_0}{T'^2}V_0^2(X' + 2\ln T'). \tag{3.19}$$

Now considering the expression for $T_{\mu\nu}T^{\mu\nu}$ we see this will diverge as $T' \to 0$ unless either

$$\Psi'_0(X') - \frac{T'_0}{2}V_0(X') = 0 \tag{3.20}$$

or

$$V_0(X' + 2\ln T') = O(T'). \tag{3.21}$$

We can integrate (3.20) with respect to X' to obtain

$$\int_{X'}^{X'+2\ln T'} \Psi'_0(w)dw = \frac{T'_0}{2}\int_{X'}^{X'+2\ln T'} V_0(w)dw$$

$$\Rightarrow \Psi_0(X' + 2\ln T') = \Psi_0(X') + \frac{T_0'}{2}\int_{X'}^{X'+2\ln T'} V_0(w)dw. \tag{3.22}$$

Using (3.13) this gives

$$\Phi(T', X') = \Psi_0(X' + 2\ln T'). \tag{3.23}$$

On the other hand (3.21) states that $V_0(v) \to 0$ as $T' \to 0$ where $v = X' + 2\ln T'$. Now $V_0(X')$ is part of the initial data given on $T' = T_0'$. $V_0(X')$ will be a continuous periodic function so that the data at $X' = 0$ agrees with the data at $X' = 2\pi$. Hence the only way that $V_0(X' + 2\ln T')$ can go to zero as T' approach zero for any value of X' is if $V_0(X')$ vanishes for all X'. Hence by (3.13)

$$\Phi(T', X') = \Psi_0(X'). \tag{3.24}$$

So for the special cases $\Phi(T', X') = \Psi_0(X' + 2\ln T')$ and $\Phi(T', X') = \Psi(X')$, and only for these, we find that $T_{\mu\nu}T^{\mu\nu}$ is finite, and no singularity forms. However, generic initial data given on a $T' = constant$ hypersurface will evolve in such a way that results in a singularity at the chronology horizon ($T' = 0$). So in general, an arbitrarily weak field diverges on the chronology horizon, and the back reaction might alter the spacetime metric sufficiently to prevent the causality violating regions from forming.

How can this be explained physically? Recall the earlier description of Misner space. As a null geodesic crosses the identification point it becomes blue shifted. The frequency is boosted by a factor:

$$\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}} = \left(\frac{1+\tanh a}{1-\tanh a}\right)^{\frac{1}{2}} \tag{3.25}$$

$$= e^a.$$

Misner space has two sets of null geodesics. One set of the geodesics works its way up to the chronology horizon, spiralling round and round the spacetime as it does so. Each time a geodesic circles the spacetime and passes through the identification point it gets blue shifted by e^a. Geodesics spiral round an infinite number of times, never quite reaching the horizon, but getting closer on each trip. So we have a field that is piling up on the horizon, getting more and more blue shifted as it does so. As the geodesics spiral round an infinite number of times they will get infinitely blue shifted. Such a field will give rise to a divergent stress-energy tensor, presumably preventing the CTCs from evolving.

The reason that fields with $\Phi = \Psi_0(X')$ do not de-stabilise Misner space is that they represent waves propagating along the other set of null geodesics that pass straight through the chronology horizon without spiralling/blue shifting. (Fields with $\Phi = \Psi_0(X' + 2\ln T')$ are also finite as there exists a coordinate transformation that swaps over the roles of the two sets of null geodesics in Misner space.)

The above divergent behaviour can also be made clear by going to the covering space. The fundamental solution to the wave equation in two dimensional Minkowski space is given by

$$\Phi = \frac{1}{2}H(t - t_0 - |x - x_0|).$$

where H is the Heaviside step function, (t, x) is the field point and (t_0, x_0) is the source point. The fundamental solution has support inside the past light cone of

(t, x) so that if (t_0, x_0) lies in the past lightcone of (t, x) then $\Phi = \frac{1}{2}$, otherwise $\Phi = 0$. Recall that Misner space may be thought of a obtained from the covering space of 2–dimensional Minkowski space (minus the origin) by identifying under a boost. Thus given a source point at (t_0, x_0) in Misner space there exists many images of that source at the points $(t_0 \cosh a + x_0 \sinh a, x_0 \cosh a + t_0 \sinh a)$ in the covering space. Thus Φ_m,the fundamental solution of the wave equation in Misner space, has the form

$$\Phi_m = \sum_{n=-\infty}^{\infty} \frac{1}{2} H(t - t_0 \cosh na - x_0 \sinh na - |x - x_0 \cosh na - t_0 \sinh na|). \quad (3.26)$$

It is clear that as one approaches $t^2 - x^2 = 0$ (the chronology horizon) the number of image points in the past cone of (t, x) approaches infinity. Thus making Φ_m diverge at the horizon.

So from all the considerations above it appears that Misner space is classically unstable, with a test scalar field evolving in the spacetime diverging at the chronology horizon.

3.1.1 The Cauchy Problem in Gott Space and Grant Space

We will now look at the Cauchy problem in Gott space. This problem will be considered from three different points of view. The first of these involves using a Greens function method to derive a Kirchoff formula for the problem. Next we consider the problem of existence and uniqueness for solutions of the wave equation in this spacetime and introduce a (modified) Sobolev space for the initial data which guarantees a unique C^k solution of the wave equation in the entire chronal region. Finally we look at the problem from the point of view of Fourier analysis by performing a change of coordinates so that all the isometries of Gott space become manifest in one periodic coordinate.

The Greens Function Approach

Before we look at the Gott space problem let us look at how we solve the Cauchy problem in Minkowski space. The simplest problem is to solve the wave equation $\Box\Phi = 0$ subject to initial data on the surface S given by $t = 0$. We specify the following data on $t = 0$:

$$\Phi(t = 0, \mathbf{r}) = \Psi_0(\mathbf{r}), \; \left.\frac{\partial \Phi}{\partial t}\right|_{t=0} = V_0(\mathbf{r}), \tag{3.27}$$

where we are looking at the evolution of initial data to a give a solution to the wave equation for $t > 0$. For $t < 0$ we take the value of Φ to be zero (Φ is "switched on" at $t = 0$). Solving $\Box\Phi = 0$ subject to initial data on $t = 0$ is equivalent to solving the inhomogeneous wave equation

$$\Box\tilde{\Phi} = f(\mathbf{r}), \tag{3.28}$$

where $\tilde{\Phi}(t, \mathbf{r}) = H(t)\Phi(t, \mathbf{r})$ and $f(\mathbf{r})$ is given by

$$f(\mathbf{r}) = \delta'(t)\Psi_0(\mathbf{r}) + \delta(t)V_0(\mathbf{r}). \tag{3.29}$$

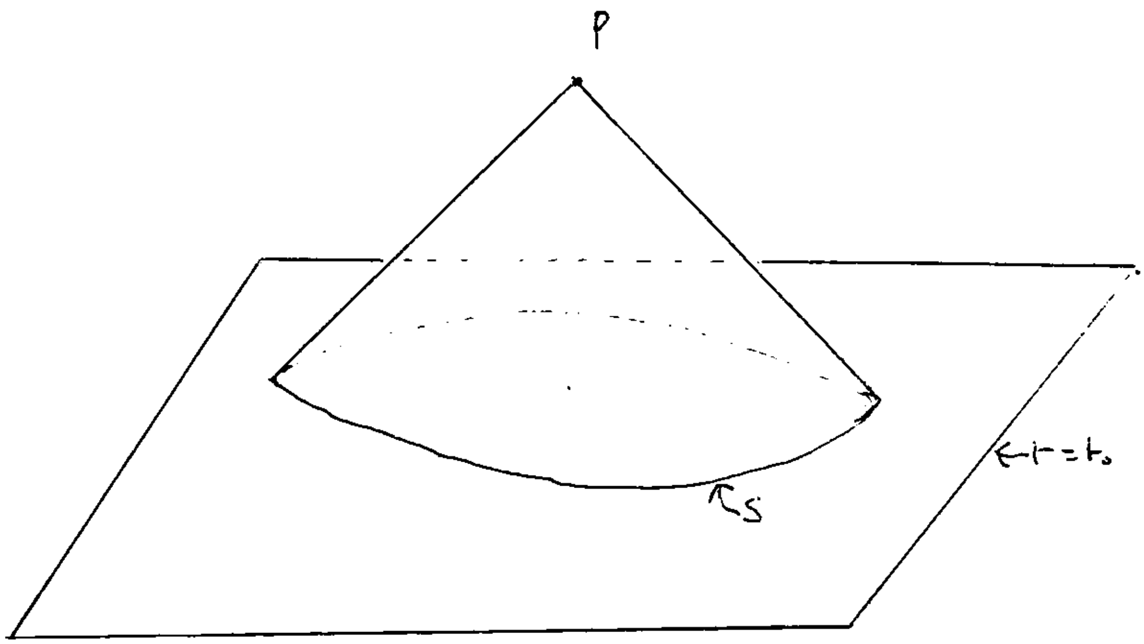

Figure 3.1: *The Intersection of the past light cone of P with the surface $t = t_0$*

We can solve the inhomogeneous wave equation by using the Greens function. For the wave equation in Minkowski space the appropriate Greens function is

$$\frac{\delta(t - t_0 - |\mathbf{r} - \mathbf{r}_0|)}{|\mathbf{r} - \mathbf{r}_0|}. \tag{3.30}$$

(3.30) is the causal, retarded Greens function for a source at $(t_0, \mathbf{r_0})$ with field point $(t, \mathbf{r})$. It has support for all $(t_0, \mathbf{r_0})$ that lie on the past null cone of $(t, \mathbf{r})$. So we get

$$\Phi(t, \mathbf{r}) = \int dt_0 \int dv_0 [\delta'(t_0)\Psi_0(\mathbf{r}_0) + \delta(t_0)V_0(\mathbf{r}_0)] \frac{\delta(t - t_0 - |\mathbf{r} - \mathbf{r}_0|)}{|r - r_0|}. \tag{3.31}$$

In (3.31) dv_0 is the surface element on S, and r_0 is a point in the Cauchy surface.

As can be seen in fig (3.1) we pick up data from the surface S that is the intersection of our past light come with the Cauchy surface. In this case S is a two-sphere (see fig (3.1)).

Using:

$$\int f(x)\delta'(x-y)dx = -f'(y), \tag{3.32}$$

$$\int f(x)\delta(x-y)dx = f(y), \tag{3.33}$$

we get

$$\begin{aligned}\Phi(t,r) &= -\frac{\partial}{\partial t_0}\int dv_0\Psi_0(\mathbf{r}_0)\frac{\delta(t-|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|}+\int dv_0 V_0(\mathbf{r}_0)\frac{\delta(t-|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|}\\ &= \frac{\partial}{\partial t}\int dv_0\Psi_o(\mathbf{r}_0)\frac{\delta(t-|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|}+\int dv_0 V_0(\mathbf{r}_0)\frac{\delta(t-|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|}. \end{aligned}\tag{3.34}$$

Now taking $\mathbf{r} = (x, y, x)$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$ the argument of $\delta(t - |\mathbf{r} - \mathbf{r}_0|)$ vanishes when

$$t = ((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{\frac{1}{2}}. \tag{3.35}$$

Defining polar coordinates by

$$\begin{aligned} x - x_0 &= t\cos\theta,\\ y - y_0 &= t\sin\theta\sin\phi,\\ z - z_0 &= t\sin\theta\cos\phi. \end{aligned}\tag{3.36}$$

We can see that the integral in (3.34) is essentially reduced to a surface integral of the initial data over a sphere of radius t. On performing the $r-$integration dv_0 reduces to $dv_0 = t\sin\theta d\theta d\phi$. We therefore obtain

$$\begin{aligned}\Phi(t,\mathbf{r}) &= \frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} t\Psi_0(x-t\cos\theta, y-t\sin\theta\sin\phi, z-t\sin\theta\cos\phi)\sin\theta d\theta d\phi\\ &+ \int_0^{2\pi}\int_0^{\pi} tV_0(x-t\cos\theta, y-t\sin\theta\sin\phi, z-t\sin\theta\cos\phi)\sin\theta d\theta d\phi. \end{aligned}\tag{3.37}$$

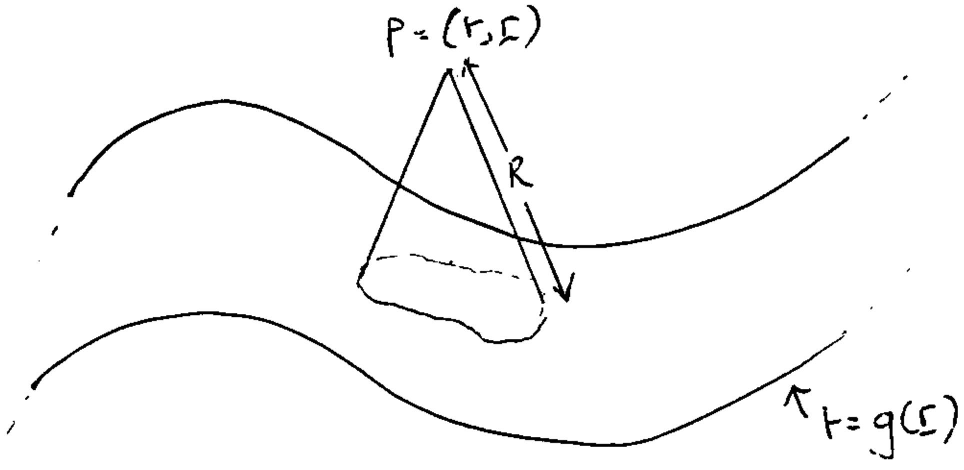

Figure 3.2: *The intersection of the the past light cone of P with the surface* $t = g(\mathbf{r})$.

(3.37) is the **Kirchoff formula** for the solution to the wave equation given initial data given on $t = constant$.

We now consider data on a more general surface $t = g(\mathbf{r})$ in Minkowski space. We will use ideas from the above to solve the Cauchy problem in this more general case. We again solve an inhomogeneous wave equation $\Box\tilde{\Phi} = f(t, \mathbf{r})$, where this time

$$\Phi(t, \mathbf{r}) = H(t - g(\mathbf{r}))\tilde{\Phi}(t, \mathbf{r}). \tag{3.38}$$

To find the appropriate f we define new coordinates:

$$T = t - g(\mathbf{r}),\ X = x,\ Y = y,\ Z = z, \tag{3.39}$$

where (t, x, y, z) are the usual Minkowski coordinates. We re-write the wave operator in these new coordinates:

$$\Box'\Phi = \frac{\partial^2\Phi}{\partial T^2} - \sum_{i=1}^{3}\frac{\partial^2\Phi}{\partial(X^i)^2} + 2(\nabla g).(\nabla\Phi_{,T}) + (\nabla^2 g)\frac{\partial\Phi}{\partial t} - (\nabla g).(\nabla g)\frac{\partial^2\Phi}{\partial T^2} = 0 \tag{3.40}$$

Now consider putting initial data on the surface S given by $T = t - g(\mathbf{r}) = 0$. In these coordinates $\widetilde{\Phi} = H(T)\Phi(T, X, Y, Z)$, and substituting $\widetilde{\Phi}$ into the wave equation we get:

$$\begin{aligned}\Box'\Phi &= \left[(1 - \nabla g.\nabla g)\left.\frac{\partial\Phi}{\partial T}\right|_S + 2(\nabla g).(\nabla\Phi)|_S + (\nabla^2 g)\Phi|_S\right]\delta(T) \\ &+ \left[(1 - (\nabla g).(\nabla g))\Phi|_S\right]\delta'(T) \qquad (3.41) \\ &= \Phi_0\delta'(T) + \delta(T)\Phi_1,\end{aligned}$$

where,

$$\Phi_0 = (1 - (\nabla g).(\nabla g))\Phi|_S, \qquad (3.42)$$

$$\Phi_1 = (1 - \nabla g.\nabla g)\left.\frac{\partial\Phi}{\partial T}\right|_S + 2(\nabla g).(\nabla\Phi)|_S + (\nabla^2 g)\Phi|_S. \qquad (3.43)$$

Hence

$$\begin{aligned}f(T, \mathbf{R}) &= \delta'(T)(1 - (\nabla g).(\nabla g))\Phi|_S \qquad (3.44) \\ &+ \delta(T)(1 - \nabla g.\nabla g)\left.\frac{\partial\Phi}{\partial T}\right|_S + 2(\nabla g).(\nabla\Phi)|_S + (\nabla^2 g)\Phi|_S.\end{aligned}$$

To find Φ we first return to Minkowski space coordinates and solve $\Box\Phi = f$ using the Minkowskian Greens function. This gives:

$$\Phi(t, \mathbf{r}) = \int \frac{\delta(t - t' - |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} f(t', \mathbf{r}') d^4x'. \qquad (3.45)$$

To calculate the integral we convert back from the $(t', \mathbf{r}')$ coordinates to the $(T', \mathbf{R}')$ coordinates, where:

$$\mathbf{R}' = (X', Y', Z'). \qquad (3.46)$$

The Jacobian for this coordinate change is equal to one so we get

$$\phi(t,\mathbf{r}) = \int \frac{\delta(t - T' - g(\mathbf{R}') - |\mathbf{r}-\mathbf{R}'|)}{|\mathbf{r}-\mathbf{R}'|} f(T',\mathbf{R}')dT'd\mathbf{R}'. \tag{3.47}$$

Using the above form of f we then get (on dropping the primes):

$$\Phi(t,x,y,z) = \int \frac{\delta(t - T - g(\mathbf{R}) - |\mathbf{r}-\mathbf{R}|)}{|\mathbf{r}-\mathbf{R}|} \left(\delta'(T)\Phi_0 + \delta(T)\Phi_1\right) dTd\mathbf{R}. \tag{3.48}$$

Performing the T integral gives:

$$\begin{aligned}\Phi(t,x,y,z) &= \int_S \frac{\delta(t - g(\mathbf{R}) - |\mathbf{r}-\mathbf{R}|)}{|\mathbf{r}-\mathbf{R}|}\Phi_1|_{T=0}d\mathbf{R} \\ &+ \frac{\partial}{\partial t}\int_S \frac{\delta(t - g(R) - |\mathbf{r}-\mathbf{R}|)}{|\mathbf{r}-\mathbf{R}|}\Phi_0|_{T=0}d\mathbf{R}\end{aligned} \tag{3.49}$$

To evaluate this integral we convert to polar coordinates (as in the first example):

$$\begin{aligned} x - X &= R\cos\theta \\ y - Y &= R\sin\theta\cos\phi \\ z - Z &= R\sin\theta\sin\phi \end{aligned} \tag{3.50}$$

Under this coordinate change $dXdYdZ = R^2\sin\theta drd\theta d\phi$. Now the argument of the delta function is zero when $R = t - g(\mathbf{R})$, so:

$$\begin{aligned}\Phi(t,x,y,z) &= \int \Phi_1(x - R\cos\theta, y - R\sin\theta\cos\phi, z - R\sin\theta\sin\phi)R\sin\theta d\theta d\phi \\ &+ \frac{\partial}{\partial t}\int \Phi_0(x - R\cos\theta, y - R\sin\theta\cos\phi, z - R\sin\theta\sin\phi)R\sin\theta d\theta d\phi\end{aligned} \tag{3.51}$$

where R is (up to a factor of $\sqrt{2}$) the distance down the past lightcone of $P = (t,x,y,z)$ to the Cauchy surface S. R is a function of θ and ϕ. Note that the integral is taken over the intersection of the past null cone with the surface S.

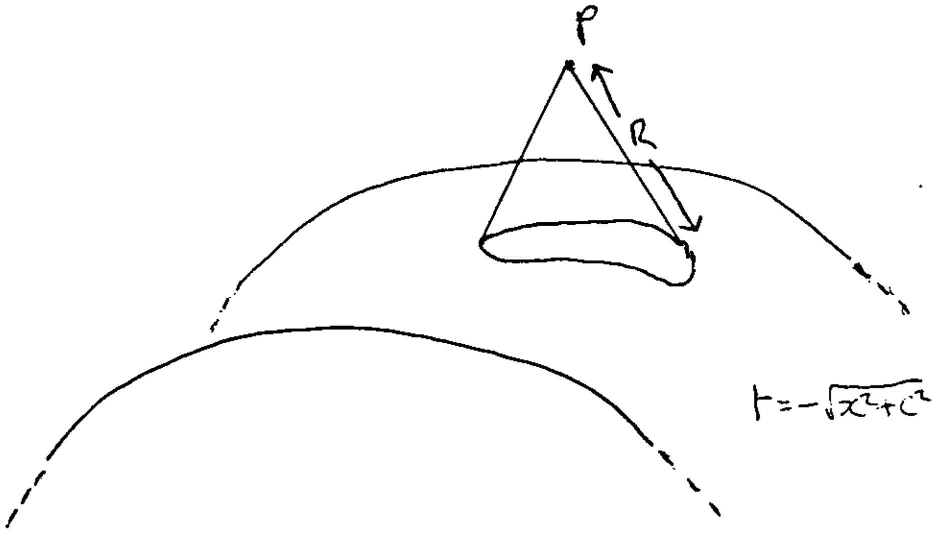

Figure 3.3: *The Cauchy surface* $t = -\sqrt{C^2 + x^2}$.

When we come to Gott space we will be interested in evolving data in the globally hyperbolic region (that is the region $t^2 - x^2 > 0$ and $t < 0$). Suitable Cauchy surfaces for this region are given the (past branch of) hyperboloids $t^2 - x^2 = C^2$, so we take S to be the surface $t = -\sqrt{x^2 + C^2}$ which gives $g(\mathbf{R}) = -\sqrt{x^2 + C^2}$ (see fig(3.3) . With this choice Φ_0 and Φ_1 are given by:

$$\begin{aligned}
\Phi_0 &= \frac{C^2}{C^2 + X^2}\Phi|_S, \qquad (3.52)\\
&= \frac{C^2}{C^2 + x^2}\Phi|_S \\
\Phi_1 &= \frac{C^2}{C^2 + X^2}\left.\frac{\partial \Phi}{\partial T}\right|_S - \frac{C^2}{(\sqrt{C^2 + X^2})^3}\Phi|_S - 2\frac{X}{\sqrt{C^2 + X^2}}\left.\frac{\partial \Phi}{\partial X}\right|_S \\
&= \left[\frac{\partial \Phi}{\partial t} - \frac{x}{C^2 + x^2}\frac{\partial \Phi}{\partial x}\right]_S + \frac{C^2}{(\sqrt{C^2 + x^2})^3}\Phi|_S \qquad (3.53)\\
&+ \left[\frac{x^2}{C^2 + x^2}\frac{\partial \Phi}{\partial t} - \frac{x}{\sqrt{x^2 + C^2}}\frac{\partial \Phi}{\partial x}\right]_S .
\end{aligned}$$

We can also calculate R for this surface. To find R we simply trace back a geodesic from (t, x, y, z) and see where it hits the surface S. That is, solve

$$t - R + \sqrt{(x - R\cos\theta) + C^2} = 0 \tag{3.54}$$

for R. Doing so gives us the quadratic

$$\sin^2\theta R^2 + 2(x\cos\theta - t)R + t^2 - x^2 + C^2 = 0, \tag{3.55}$$

and hence

$$R = \frac{t - x\cos\theta + \sqrt{(x - t\cos\theta)^2 + C^2\sin^2\theta}}{\sin^2\theta}. \tag{3.56}$$

The plus sign is taken to ensure R is positive. Multiplying both sides of (3.56) by $\sin^2\theta$ and putting the form for $R\sin^2\theta$ into (3.55) gives:

$$R = \frac{C^2 - (t^2 - x^2)}{x\cos\theta - t + \sqrt{(x - t\cos\theta)^2 + C^2\sin^2\theta}}. \tag{3.57}$$

By substituting for R, Φ_0 and Φ_1 in (3.51) we have derived the Kirchoff formula for the solution to the Cauchy problem in Minkowski space with initial data given on the Cauchy surface $t = -\sqrt{x^2 + C^2}$.

This solution is written in terms of the integral (3.51), so as long as this integral converges we have a finite solution to the wave equation. Instead of thinking of (3.51) as an integral of $R\Phi_0$ and $R\Phi_1$ over the intersection of the past null cone of the field point with the Cauchy surface we may regard (3.51) as an integral of $R\Phi_0$ and $R\Phi_1$ over the past celestial sphere. As long as $R\Phi_0$ and $R\Phi_1$ are finite everywhere then the integral in (3.51) will converge, giving a finite solution to the wave equation. It should be noted however that the integral can still converge even if the integrand diverges. As we approach the Cauchy horizon $x - t = 0, x + t = 0$ the value of R (3.56) becomes infinite at $\theta = 0,\ \pi$, however provided Φ_0 and Φ_1 fall off sufficiently fast as $R \to \infty$ we can still have a convergent integral. So provided our initial data falls off sufficiently fast in the $t,\ x$ directions there does indeed exist a solution to the wave equation which remains finite on the horizon. For example it would be enough if Φ_0 and Φ_1 were both $O(R^{-1})$.

We will now use the Minkowski space formula to try and construct a solution to the wave equation in Gott space by reducing the Gott space problem to a Cauchy problem in Minkowski space with modified data — as will be explained below.

Before we look at the Cauchy problem in Gott space let us first consider a related, and simpler problem - the Cauchy problem for a single cosmic string with deficit angle π. Here we will use ideas from chapter two about putting a covering space on this single string spacetime. Consider putting data on a $t = constant$ Cauchy surface in the cosmic string spacetime, with the string at $x = 0, y = d$. Now recall from chapter two that when we trace back geodesics in the single string spacetime, the intersection of a past null cone with the Cauchy surface will not be a three-sphere (as in Minkowski space) but will be made up of two parts of a three-sphere each lying on either side of the cosmic string, see fig (3.4).

Now one way to modify the Kirchoff formula for Minkowski space to the cosmic string spacetime is to consider images of the field point. If we take our field point to be at $P = (t, x, y, z)$, then its image point (after rotation about the string) will be at $P' = (t, -x, 2d - y, z)$. Now about each point construct a Kirchoff type integral but

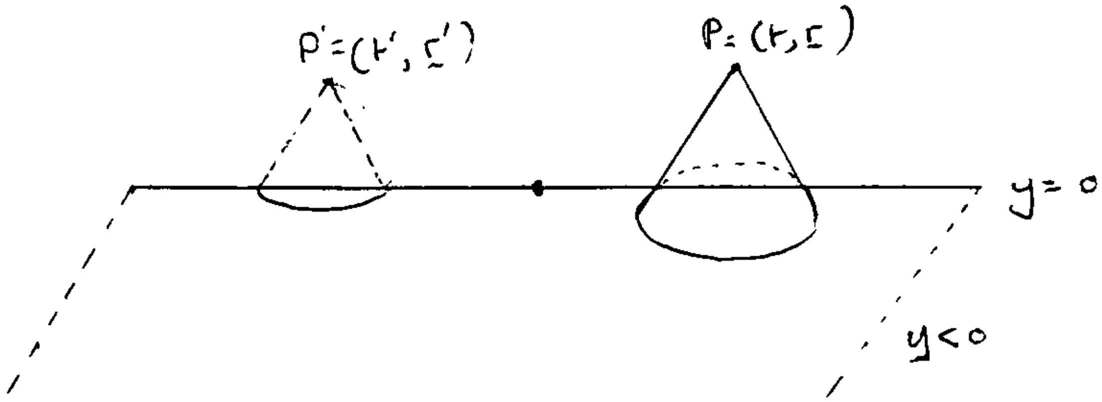

Figure 3.4: *The past light cone of a point P hitting $t = t_0$. The light cone encounters the string wedge.*

with the ranges of θ and ϕ modified to account for the fact that we only have parts of a three-sphere to integrate over. As one may expect this method can prove a bit messy, and a far more elegant approach is to consider the covering space. One can extend the data on $t = t_0$ in the physical space to the whole of the covering space. Recall that the π-wedge cosmic string spacetime is the quotient of Minkowski space under a rotation of angle π. In order for data on the covering space to represent data on the physical space we must construct it so that the data evaluated at the point (t, x, y, z), with $y > d$, is equal to the data evaluated at the point $(t, x, 2d - y, z)$. So if $\phi(t, x, y, z)$ represents the data on the physical space we extend it to the covering space in a fashion such that:

$$\phi(t, x, y, z) = \phi(t, -x, 2d - y, z) \tag{3.58}$$

That is, if A is a member of the isometry group on the covering space then a function

Φ_C on the covering space represents a function Φ_p on the physical spacetime if at any point $\mathbf{p}$:

$$\Phi_C(A^n\mathbf{p}) = \Phi_P(\mathbf{p}). \tag{3.59}$$

In particular notice that

$$\phi(t, x, y = d, z) = \phi(t, -x, y = d, z), \tag{3.60}$$

so the data is continuous across the string. We now solve the Cauchy problem on the covering space using the usual Minkowskian Greens function. Because of our choice of data the solution is also invariant under the appropriate isometry and hence defines a solution on the physical space. In the above we have turned the Cauchy problem in the single string spacetime into a Cauchy problem in Minkowski spacetime with data that satisfies certain periodicity relations.

We now consider the problem in Gott space. In order to get a Kirchoff formula in Gott space we need a knowledge of the past light cone of a field point in the Gott space. To do this we trace back null geodesics from the field point and look to see where they hit the Cauchy surface $(for - d \leq y \leq d)$. This is equivalent to tracing back null geodesics from images of the field point, and seeing where the hit the Cauchy surface $(-d \leq y \leq d)$. The intersection of each image point's past null cone with the Cauchy surface will be a fraction of a "sphere" (in fact it will not be a metric sphere with the Cauchy surface we will use). The Kirchoff formula will then consist of a sum of integrals over these partial "spheres". Now one may suspect that if the chronology horizon of Gott space is to be unstable it will be due to the fact that as we approach the horizon an infinite number of image points have light cones that intersect the Cauchy surface (as is the case for Misner space) , and when we then write down our Kirchoff integral it will then consist of an infinite sum of contributions from these image points. Is this in fact the case? Consider a field point at (T, X, Y, Z) and its n^{th} image point given by $(T \cosh 4na - X \sinh 4na, X \cosh 4na - T \sinh 4na, Y - 4nd, Z)$.

Gott space is uniform in the z direction, so in what follows we will ignore the physically uninteresting z direction and do all our calculations in the $\phi = 0$ plane.

To see where the past light cone of the field point hits $t^2 - x^2 = C^2$ we trace back geodesic from images of the field point and see where they hit $t^2 - x^2 = C^2$. As already stated, tracing back geodesics from the n^{th} image point (T_n, X_n, Y_n) is equivalent to tracing back geodesics from the field point which wind n times round both strings. Let θ label the geodesic and R be an affine parameter, then a point on the geodesic is given by $(t(R), x(R), y(R))$ where

$$\begin{aligned} t(R) &= T_n - R \\ x(R) &= X_n - R\cos\theta \\ y(R) &= Y_n + R\sin\theta \end{aligned} \tag{3.61}$$

The geodesics hit the Cauchy surface $t^2 - x^2 = C^2$ when

$$R = \frac{T_n - X_n\cos\theta + \sqrt{(X_n - T_n\cos\theta)^2 + C^2\sin^2\theta}}{\sin^2\theta}. \tag{3.62}$$

We require the geodesics to hit the Cauchy surface for $-d \leq y \leq d$, so we require

$$(4n-1)d - Y \leq R\sin\theta \leq (4n+1)d - Y, \tag{3.63}$$

giving

$$(d+Y)+R\sin\theta \geq 4nd \geq (Y-d)+R\sin\theta. \tag{3.64}$$

The maximum value of n is when $R\sin\theta$ is at its largest. Differentiating $R\sin\theta$ with respect to θ we find it has a maximum at $\cos\theta = \frac{X}{T}$ giving

$$R\sin\theta = C - \sqrt{T^2 - X^2}. \tag{3.65}$$

Which gives a maximum value of n for given (T, X, Y) as

$$n = \frac{d + Y + C - \sqrt{T^2 - X^2}}{4d}. \tag{3.66}$$

The maximum value of this can attain for an arbitrary field point in the chronal region is given by the integer part of

$$\frac{d + C + Y}{2d}. \tag{3.67}$$

So we see that in fact geodesics only spiral round the spacetime a finite number of times as they leave the Cauchy surface and work their way up to the field point (T, X, Y), even when the field point is located near the chronology horizon. This is in contrast to what happens in Misner space and is due to the fact that points are translated in the y direction when identified. So there will only be a finite number of contributions to the Kirchoff integral from each image point. This indicates that the Gott space instability (if indeed it is unstable) may be considerably weaker than that for Misner space.

As we saw in the case of the single cosmic string, considering image points can prove a bit messy. Instead of thinking about images of the field point we will look at the problem from the point of view of the covering space. This method involves putting data on a Cauchy surface in the physical Gott space ($-d \leq y \leq d$), and then extending this data to the covering space in a suitable way. Once we have constructed the data on the covering space we are left with a "simple" Cauchy problem in Minkowski space.

We now show how to construct such data. First put data on the physical Cauchy surface and we now regard this as data on the portion of the covering space with $-d \leq y \leq d$. We now extend this data throughout the rest of the covering spacetime in an appropriate way. Gott space is the quotient of Minkowski under the group of isometries generated by C_1 and C_2, where C_1 maps (t, x, y) to the point

$$(t \cosh 4a - x \sinh 4a, x \cosh 4a - t \sinh 4a, y - 4nd), \tag{3.68}$$

and an element C_2 maps (t, x, y) to the point

$$(t \cosh 2a - x \sinh 2a, -x \cosh 2a - t \sinh 2a, -y + 2nd). \tag{3.69}$$

So a function Φ_C on the covering space represents a function Φ_P on the physical space if at any point $\mathbf{p}$ we have

$$\Phi_P(\mathbf{p}) = \Phi_C(C^m D^n \mathbf{p}). \tag{3.70}$$

We can now proceed to construct our data on the covering space. Take the data on the covering space for $-d \leq y \leq d$ and extend it for values of $y > d$ and $y < -d$ in a way consistent with (3.70). In doing so we see that the data on the covering space, denoted by Φ_d must satisfy:

$$\Phi_d(t, x, y) = \Phi_d(t \cosh 4na - x \sinh 4na, x \cosh 4na - t \sinh 4na, y - 4nd), \tag{3.71}$$

for $(4n-1)d \leq y \leq (4n+1)d$, and

$$\begin{aligned}\Phi_d(t,x,y) &= \Phi_d(t\cosh(4n+2)a - x\sinh(4n+2)a, -x\cosh(4n+2) \\ &\quad +t\sinh(4n+2)a, -y+(4n+2)d),\end{aligned}$$

for $(4n+1)d \leq y \leq (4n+3)d$.

So we see that data in the covering space must be invariant under the above boosts and translations, as was indicated from forming the covering space. Now that we have our adjusted data we integrate it using the usual Minkowski Greens function to obtain a solution on the covering space. Because of the way we have constructed our data the solution is automatically invariant under the isometry group and hence represents a solution on the physical space obtained by taking the quotient.

From the above calculation we see that the t and x behaviour of our data depends on the value of n which in turn depends on y. The Cauchy surface is divided up into strips in the y-direction, each strip having a different value of n corresponding to the number of times the geodesic wraps round the strings in the physical space, see figure (3.5) As we go further out in the y-direction we pick up more strips. Now recall that the maximum value of $R\sin\phi$ is $C - \sqrt{T^2 - X^2}$ at $\cos\phi = \frac{X}{T}$. Let S denote the surface formed by the intersection of the past light cone of (T, X, Y) with the Cauchy surface. The value of y for the surface S ranges between:

$$Y - C + \sqrt{T^2 - X^2} \leq y \leq Y + C - \sqrt{T^2 - X^2}. \tag{3.72}$$

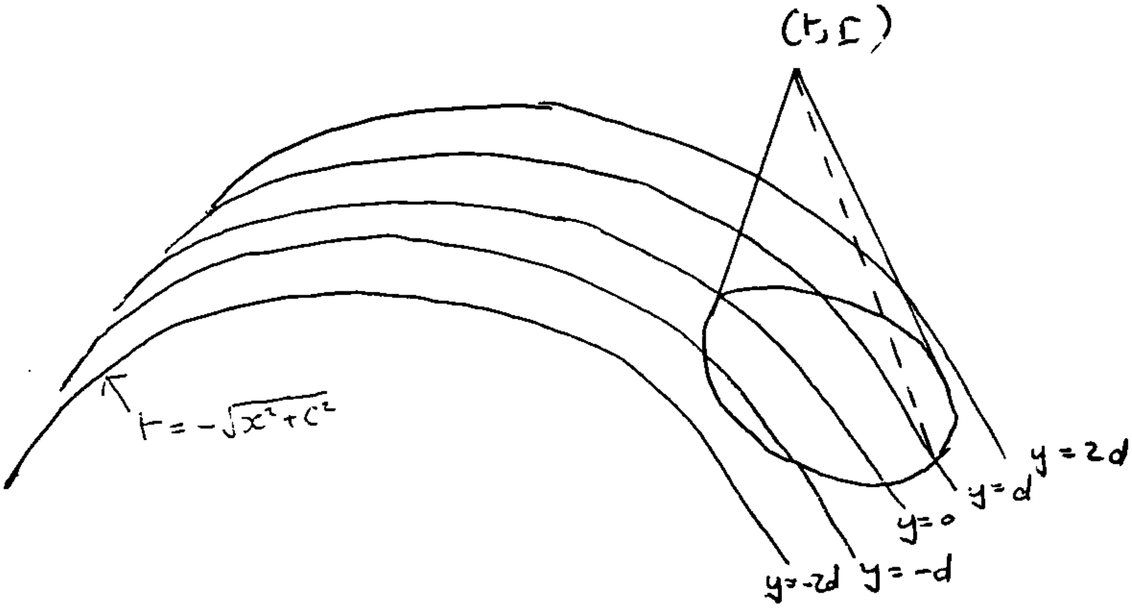

Figure 3.5: *The Cauchy surface in the covering space divided into strips in the y-direction, Here we get contributions from geodesics that have travelled once via the top/bottom strings.*

As we approach the chronology horizon we get:

$$Y - C \leq y \leq Y + C. \tag{3.73}$$

So the maximum number of strips we will pick up will be the integer part of

$$\frac{d + C + Y}{2d}. \tag{3.74}$$

So that we will get contributions from geodesics that have spiralled at most

$$\left[\frac{d + C + Y}{4d}\right], \tag{3.75}$$

times round both strings. This is in agreement with the number obtained from the image point point of view.

It seems almost obvious that given the above we can also construct a finite solution the wave equation on Gott space. Take data that satisfies the fall off conditions in t, x, discussed previously when we considered solutions to the problem in Minkowski space, and put it on the Cauchy surface in Gott space ($-d \leq y \leq d$). Now translate this data to the covering space via the Gott space isometries in a fashion already discussed. As $R\Phi_0$ and $R\Phi_1$ are finite everywhere on the strip $-d \leq y \leq d$ (by construction) then they will be finite on the all the strips (as the data is simply translated in the t, x direction in going from one strip to another). As we only pick up a finite number of strips in our integration, and because $R\Phi_0$ and $R\Phi_1$ are finite everywhere this data will also evolve to a finite solution to the wave equation.

In the above we see that we can in fact find finite solutions, Φ, to the wave equation on Gott space. However we made a rather unnatural mathematical restriction on the data, rather than physically reasonable assumptions such as requiring the energy to be finite. Furthermore we simply looked at the finiteness of Φ and did not look at the physical properties of the solution. Physically it would be natural to demand that our data has finite energy on the Cauchy surface (for example, in the Minkowski space case this would imply putting some fall off conditions in the y-direction) , and that it evolves into a finite energy solution of the wave equation. Also, if we require the stress-energy tensor for our scalar field to be finite we need pointwise convergence of first derivatives of Φ as well as pointwise convergence of Φ. These problems (along with others) will be discussed in the next section.

3.1.2 Is The Cauchy Problem in Gott Space Well Posed?

In this section we return to the Cauchy problem in Gott space and consider suitable function spaces for the initial data which ensure the solution is well behaved up to (and including) the chronology horizon. We first consider the initial value problem for the wave equation on the region $t^2 - x^2 > 0$ of three-dimensional Minkowski space given initial data on the past hyperboloid S_{τ_0} given by $t^2 - x^2 = \tau_0^2$. We put initial data, in the form of the value of a field Φ and its normal derivative on the spacelike surface S_{τ_0}. There should be a unique solution to the wave equation with this initial data. It is not difficult to show that this is the case when one considers analytic initial data. This can be see as follows. From the initial data one can compute all tangential derivatives of Φ and its normal derivatives on the initial data surface. From the wave equation we can get second order normal derivatives of Φ, and differentiating the wave equation we can get all the tangential derivatives of the second order normal derivatives. We may then differentiate the wave equation to find third order normal derivatives, and proceed to find all the tangential derivatives of these. We can continue this procedure to find all the derivatives of Φ and so write down a formal power series for a solution of Φ in a neighbourhood of the initial data surface. The convergence of this series is proven in the Cauchy-Kowalevskaya theorem which states that [CDD]: *Given analytic initial data for the wave equation on a spacelike surface S_{τ_0} there exists strictly one analytic solution in a neighbourhood* *of* S_{τ_0}.

As it stands the Cauchy-Kowalevskaya theorem only shows existence of a solution in a neighbourhood, U, of S_{τ_0}. We can extend the proof for the whole of $D^+(S_{\tau_0})$ as follows. Define local coordinates x^1, x^2, x^3 and $x^0 = \tau$ on a coordinate neighbourhood U. Let the initial data be analytic functions of x^1, x^2 and x^3 on $S_{\tau_0} \cap U$ (S_{τ_0} is the surface $\tau = \tau_0$.) Using the method described above one can construct a solution to the wave equation from a power series that converges within a neighbourhood U of the initial data surface. Now cover S_{τ_0} with coordinate neighbourhoods of the form U, and in each coordinate neighbourhood construct a solution as before. We now have a solution within a region of the surface S_{τ_0}. A consequence of the Cauchy-Kowaleskya theorem for linear differential equations with fixed coefficients [CH], such as the wave equation in Minkowski space is that the expression for the solutions converges within a fixed radius, independently of the specific initial data.

As a consequence of this we now have a solution within a fixed radius of the whole of the initial data surface. This will now induce initial data on some surface S_{τ_1}, where $\tau_1 > \tau_0$. Now repeat the above procedure for the surface S_{τ_1}, and repeating this a finite number of times we obtain a solution for the whole region $D^+(S_{\tau_0})$.

Although the Cauchy-Kowalevskaya theorem proves uniqueness and existence for analytic data it does not show the Cauchy problem is well posed for general data, nor does it say anything about the continuous dependence of the solutions on the initial data. Another important issue not dealt with by the Cauchy-Kowalevskaya theorem is that of causal propagation of the field. An analytic function is uniquely determined by values of it and its derivatives at a point, and so is uniquely determined by its value in an arbitrarily small neighbourhood of that point. If we alter the data within a neighbourhood of the initial data surface we must alter the data over the entire surface. So in order to consider causal propagation we will need to consider non-analytic initial data.

In the previous section we looked at how we might go about forming a solution to the wave equation in the chronal region given initial data on the surface $t = -\sqrt{x^2 + C^2}$. However, we did not consider what restrictions we need to make on our data to ensure it has finite energy, and that the integral form for the solution to the wave equation (obtained in the previous section) converges to a finite energy solution.

We will now use energy estimates for our field Φ to construct function spaces for our initial data and Φ in order to show that the Cauchy problem is well posed. We want to look at global existence of solutions to the wave equation in the past null cone of the origin given initial data on a non-compact Cauchy surface. As our Cauchy surface is non-compact we cannot use the existing theorems in Hawking and Ellis, so we will need to modify them to obtain uniform estimates. Since we are working with the wave equation on a space which is locally Minkowskian it is possible to obtain explicit values for the constants in the appropriate inequalities. We now sketch an outline of the proof.

We are solving $\Box\Phi = 0$ given initial data on some spacelike hypersurface given by

$f(t, x, y, z) = 0$ with unit normal vector $\mathbf{n}$. The initial data takes the form

$$\Phi_0 = \Phi|_{f=0}, \;\; \Phi_1 = \Phi,_{\mathbf{n}}|_{f=0}. \tag{3.76}$$

where

$$\Phi,_{\mathbf{n}} = n^a \Phi,_a \,. \tag{3.77}$$

(Note: these Φ_0 and Φ_1 are completely different from the Φ_0 and Φ_1 we considered in the section on the Kirchoff formula.)

It will be shown that if Φ_0 and Φ_1 lie in certain function spaces on the hypersurface, based on the physical energy of the field Φ, (that will turn out to be weighted Sobolev spaces) then Φ lies in a standard spacetime Sobolev space. Approximating our initial data by analytic fields we will show that the solutions obtained converge to a solution of the wave equation. Using the Sobolev embedding theorem we will then be able to show that Φ is pointwise convergent in the whole domain of dependence of $f(t, x, y) = 0$ including the chronology horizon. This will, as it stands, prove global existence of solutions in Minkowski space. In order to prove existence of solutions in Gott space we will need to pass to a covering space and modify the function spaces we have obtained for Minkowski space.

We will show that the Cauchy problem for the wave equation in Gott space given initial data on the spacelike surface $t = -\sqrt{x^2 + C^2}$ is well posed. Furthermore the solution remains finite at the Chronology horizon. The proof will take the form of various lemmas.

We will now proceed to construct the various functions spaces required for the existence proof. The first function space we will require is a Sobolev space. Let $\Omega \subset \mathbf{M}$ and $m \in \mathbf{Z}^{+}$. The *Sobolev space* $W^{m}(\Omega)$ is a vector space of functions Φ whose values and derivatives are defined almost everywhere on Ω and for which $\|\Phi, \Omega\|_m$ is finite where $\|\Phi, \Omega\|_m$ is defined to be

$$\|\Phi, \Omega\|_m = \sum_{p=o}^{m} \int_{\Omega} (|D^p \Phi|)^2 dV, \tag{3.78}$$

where dV is the volume element on Ω and the derivatives are taken in time and space directions with D^p given by

$$D^p \Phi = \partial_{x_1}^{p_1} \partial_{x_2}^{p_2} \partial_{x_3}^{p_3} \partial_{x_0}^{p_0} \Phi. \tag{3.79}$$

where $p_i \in \mathbf{N}$ and $p_1 + p_2 + p_3 + p_4 = p$. We will also need the *Sobolev embedding theorem* [HE]

The Sobolev Embedding Theorem

There exists a positive constant K_1 such that for any $\Phi \in W^m(\Omega)$ with $2m > n$, where n is the dimension of Ω, $|\Phi| \leq K_1 \|\Phi, \Omega\|_m$ on Ω.

From this and the fact that the vector space of all continuous functions Φ on Ω is a Banach space with norm $\sup_{x \in \Omega} |\Phi|$, it follows that if $\Phi \in W^m(\Omega)$ where $2m > n$, then Φ is continuous on Ω. Similarly if $\Phi \in W^{m+p}(\Omega)$, then Φ is C^p on Ω.

We first show that provided Φ satisfies appropriate fall off behaviour at infinity the the total energy, $E(\Phi)$ of the field Φ is conserved. We first prove some technical lemmas.

Lemma 3.1 *Let*

$$\mathbf{n} = \frac{\partial}{\partial n} = \alpha\frac{\partial}{\partial t} + \beta^i\frac{\partial}{\partial x^i} \tag{3.80}$$

be the unit normal vector to the surface $f(t,x,y) = 0$ *and let* $\mathbf{t} = \frac{\partial}{\partial t}$ *be a timelike vector. Also let* T_{ab} *be the stress-energy tensor for a scalar field* Φ *satisfying* $\Box\Phi = 0$, *given by*

$$T_{ab} = \Phi_{,a}\Phi_{,b} - \frac{1}{2}g_{ab}g^{cd}\Phi_{,c}\Phi_{,d}\,. \tag{3.81}$$

Then:

$$T_{ab}t^a n^b = (2\alpha)^{-1}\left[\Phi^2{}_{,t} + \sum_{i=1}^{3}\left(\beta^i\frac{\partial\Phi}{\partial t} + \alpha\frac{\partial\Phi}{\partial x^i}\right)^2\right]. \tag{3.82}$$

Note that $\tau^i = \beta^i\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x^i}$ *are unit spacelike vectors orthogonal to* $\mathbf{n}$.

Proof

First note $1 = g_{ab}n^a n^b = \alpha^2 - \delta_{ij}\beta^i\beta^j$, and so $\alpha^2 = 1 + \delta_{ij}\beta^i\beta^j$. Now

$$\begin{aligned}
T_{ab}t^a n^b &= \alpha t_{00} + \beta^i T_{0i} \\
&= \frac{\alpha}{2}\left[\Phi^2{}_{,t} + \Phi^2{}_{,x} + \Phi^2{}_{,y} + \Phi^2{}_{,z}\right] + \beta^i\Phi_{,t}\Phi_{,i} \\
&= \frac{1}{2\alpha}\left[\alpha^2\Phi^2{}_{,t} + 2\alpha\beta^i\Phi_{,t}\Phi_{,i} + \alpha^2\delta^{ij}\Phi_{,i}\Phi_{,j}\right] \\
&= \frac{1}{2\alpha}\left[\Phi^2{}_{,t} + \delta_{ij}\beta^i\beta^j\Phi^2{}_{,t} + 2\alpha\beta^i\Phi_{,t}\Phi_{,i} + \alpha^2\delta^{ij}\Phi_{,i}\Phi_{,j}\right] \\
&= \frac{1}{2\alpha}\left[\Phi^2{}_{,t} + \sum_{i=1}^{3}\left(\beta^i\Phi_{,t} + \alpha\Phi_{,i}\right)^2\right]
\end{aligned}$$

□

Corollary 3.2

$$\frac{1}{2g_{ab}t^a n^b}\Phi^2{}_{,t} \le T_{ab}t^a n^b. \tag{3.83}$$

Proof

$$T_{ab}t^a n^b = \frac{1}{2\alpha}\left[(\Phi^2{}_{,t} + \sum_{i=1}^{3}(\beta^i\Phi_{,t} + \alpha\Phi_{,i})^2\right]$$
$$\geq \frac{1}{2\alpha}\Phi^2{}_{,t},$$

where $g_{ab}t^a n^b = \alpha$.

□

We will prove the following proposition in the case relevant to our problem, where S_τ is given by, $\tau^2 = t^2 - x^2$, $t < 0$.

Proposition 3.3 *$E(\tau) = \int_{S_\tau} T_{ab}t^a n^b dS$ is conserved for all values of τ provided that $e^{\frac{|\xi|}{2}}\Phi = O(|\xi|^{-\frac{\alpha}{2}})$, where ξ is defined by*

$$t = \tau\cosh\xi, \;\; x = \tau\sinh\xi. \tag{3.84}$$

Proof

Let $E_b = t^a T_{ab}$ and hence,

$$E^b{}_{;b} = \frac{1}{2}T^{ab}(t_{a;b} + t_{b;a}) \tag{3.85}$$

as $T^{ab}{}_{;b} = 0$.

Consider a compact domain k with smooth boundary ∂k then we get, by the divergence theorem,

$$\int_{\partial k} E_a n^a dS = \frac{1}{2}\int_k T^{ab}(t_{a;b} + t_{a;b})dV, \tag{3.86}$$

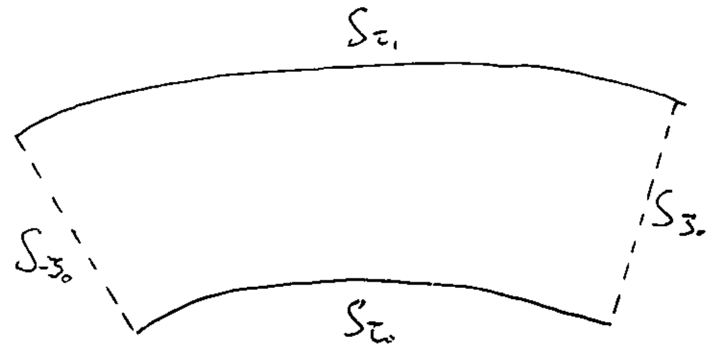

Figure 3.6: *The domain of integration D.*

where dV is the volume element of k and ds is the surface element induced on ∂k. In fact as t^a is a Killing vector we have

$$\int_{\partial k} E_a n^a dS = 0. \tag{3.87}$$

Now consider a compact domain (see fig(3.6)) bounded by the surfaces S_{τ_1}, $-\xi_0 \leq \xi \leq \xi_0$, S_{τ_0}, $-\xi_0 \leq \xi \leq \xi_0$, $S_{-\xi_0}$ and S_{ξ_0} (S_{ξ_0} is the surface $\xi = \xi_0$).

Assume that that T_{ab} is smooth and only has support in a compact region in the y and z directions.

Now from

$$\int_{\partial k} E_a n^a dS = 0, \tag{3.88}$$

we get

$$\int_{S_{\tau_1}} E_a n^a dS - \int_{S_{\tau_0}} E_a n^a dS - \int_{S_{\xi_0}} E_a n^a dS - \int_{S_{-\xi}} E_a n^a dS = 0. \tag{3.89}$$

Now consider the explicit expression for the energy on the surface S_{ξ_0}. The unit normal to this surface (in a Cartesian basis) is $\mathbf{n} = (\sinh\xi_0, -\cosh\xi_0, 0, 0)$. Written in terms of τ and ξ the energy on S_{ξ_0} is given by

$$\int_{S_{\xi_0}} \left(-\frac{1}{2}\sinh\xi_0 \left(\Phi^2{}_{,\tau} + \frac{1}{\tau^2}\Phi^2{}_{,\xi} \right) + \frac{\cosh\xi_0}{\tau}\Phi_{,\tau}\Phi_{,\xi} \right) dS. \tag{3.90}$$

If we are to have $E(\tau_1) = E(\tau_0)$ then we must have the energy of Φ on S_{ξ_0} and $S_{-\xi_0}$ falling off to zero as $\xi \to \infty$. We, therefore, demand that

$$\lim_{\xi \to \pm\infty} e^{2|\xi|}\Phi^2{}_{,\tau} = O(|\xi|^{-\alpha}), \tag{3.91}$$

$$\lim_{\xi \to \pm\infty} e^{2|\xi|}\Phi^2{}_{,\xi} = O(|\xi|^{-\alpha}), \tag{3.92}$$

where $\alpha > 0$. So if $e^{|\xi|}\Phi = O(|\xi|^{-\frac{\alpha}{2}})$ the energy on S_{ξ_0} and $S_{-\xi_0}$ will vanish as ξ goes to infinity. Hence $E(\tau_1) = E(\tau_0)$ as required.

□

Proposition 3.4 *Initial data given on a spacelike hypersurface $t = -\sqrt{x^2 + C^2}$, satisfying the fall of behaviour given in Proposition (3.3) , will evolve to give a unique solution to the wave equation.*

Proof

Consider two solutions, Φ and Ψ, to the wave equation $\Box\Phi = 0$ both evolving from the same initial data give by

$$\Phi|_{S_{\tau_0}} = \Phi_0, \Phi_{,\mathbf{n}}|_{S_{\tau_0}} = \Phi_1,$$

$$\Psi|_{S_{\tau_0}} = \Phi_0, \Psi_{,\mathbf{n}}|_{S_{\tau_0}} = \Phi_1.$$

Using conservation of energy we get

$$E(\Phi, S_{\tau_0}) = E(\Phi, S_{\tau_1}), \tag{3.93}$$

and

$$E(\Psi, S_{\tau_0}) = E(\Psi, S_{\tau_1}). \tag{3.94}$$

Now as the wave equation is linear then $\Phi - \Psi$ is also a solution with vanishing initial data. We therefore have

$$E(\Phi - \Psi, S_{\tau_0}) = 0. \tag{3.95}$$

Using conservation of energy gives

$$E(\Phi - \Psi, S_{\tau_1}) = 0. \tag{3.96}$$

Put $U = \Phi - \Psi$. As $E(U)$ vanishes on S_τ all tangential derivatives of U will vanish on S_τ (see lemma (3.1). As U is equal to zero on S_{τ_0} it will equal zero on all S_τ and hence we will have $\Psi = \Phi$ in $D^+(S_{\tau_0})$ by the Cauchy-Kowalevskaya theorem.

□

For Φ constant $E(\Phi)$ will vanish, and when Φ is not equal to a constant then $E(\Phi) > 0$. As $E(\Phi - \Psi, S_\tau)$ vanishes, then Φ and Ψ differ by, at most, a constant on S_τ. However, when $\tau = \tau_0$ this constant vanishes (as Φ and Ψ have the same initial data).

Unfortunately the energy $E(\Phi)$ only depends on derivatives of Φ and not on Φ itself. In order to prove existence we will need Φ^2 terms in our norms. We therefore modify the energy to include a Φ^2 term.

Definition 3.5 *Let $W^1_E(\Omega)$ be the first order energy-Sobolev space. It is a function space with Φ defined almost everywhere on Ω, and for which $||\Phi, \Omega||^E_1$ is finite, where*

$$||\Phi, \Omega||^E_1 = \int_\Omega [\Phi^2 g_{ab} + T_{ab}] t^a n^b dS. \tag{3.97}$$

In the above dS is the volume element on Ω.

Proposition 3.6 *Let S_{τ_0} be the initial hypersurface, S_{τ_1} the final hypersurface and V the region between them. Then if $||\Phi, S_{\tau_0}||^E_1$ is finite then $||\Phi, V||^E_1$ is finite, and moreover*

$$\int_V \Phi^2 dV$$

is finite.

Proof

$$\Phi(x, t_1) = \Phi(x, t_0) + \int_{t=t_0}^{t_1} \frac{\partial \Phi(x,t)}{\partial t} dt. \tag{3.98}$$

For fixed x_0 let t_0 be such that $\tau(x_0, t_0) = \tau_0$ and t_1 be such that $\tau(x_0, t_1) = \tau_1$. Then

$$\begin{aligned} \int_{t=t_0}^{t_1} \frac{\partial \Phi(x,t)}{\partial t} dt &= \int_{t=t_0}^{t_1} \left(g_{ab} n^a t^b\right)^{\frac{1}{2}} \left(g_{ab} n^a t^b\right)^{-\frac{1}{2}} \frac{\partial \Phi}{\partial t} dt \\ &\leq \left[\int_{t=t_0}^{t_1} g_{ab} n^a t^b dt\right]^{\frac{1}{2}} \left[\int_{t=t_0}^{t_1} \left(g_{ab} n^a t^b\right)^{-1} \left(\frac{\partial \Phi}{\partial t}\right)^2 dt\right]^{\frac{1}{2}}. \end{aligned}$$

But

$$\int_{t=t_0}^{t_1} \left(g_{ab} n^a t^b\right) dt = \int_{t=t_0}^{t_1} \alpha dt = \int_{t=t_0}^{t_1} \frac{\partial \tau}{\partial t} dt = \int_{\tau=\tau_0}^{\tau_1} d\tau = (\tau_1 - \tau_0),$$

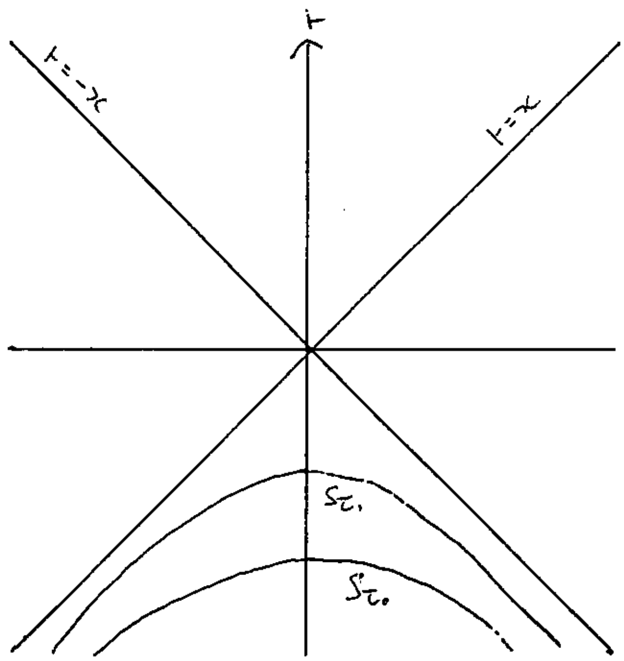

Figure 3.7: *The surfaces S_{τ_0} and S_{τ_1}.*

and by corollary (3.2)

$$\int_{t=t_0}^{t_1} \left(g_{ab}n^a t^b\right)^{-1} \left(\frac{\partial \Phi}{\partial t}\right)^2 dt \leq \int_{t=t_0}^{t_1} 2T_{ab}t^a n^b dt.$$

So we get

$$|\Phi(x,t_1)| \leq |\Phi(x,t_0)| + (\tau_1 - \tau_0)^{\frac{1}{2}} \left[\int_{t=t_0}^{t_1} 2T_{ab}t^a n^b dt\right]^{\frac{1}{2}},$$

giving

$$\Phi^2(x,t_1) \leq 2\Phi^2(x,t_0) + 4(\tau_1 - \tau_0)\int_{t=t_0}^{t_1} T_{ab}t^a n^b dt.$$

Now integrate this with respect to x, y and z, giving

$$\int_{S_{\tau_1}} \Phi^2(x,t_1(x))d^3x \leq 2\int_{S_{\tau_0}} \Phi^2(x,t_0(x))d^3x + 4(\tau_1 - \tau_0)\int_x \int_{t=t_0}^{t_1} T_{ab}t^a n^b dV,$$

where $dV = dtd^3x = dSd\tau$. Also $d^3x = g_{ab}t^a n^b dS$. So

$$\int_{S_{\tau_1}} \Phi^2 g_{ab} t^a n^b dS \leq 2\int_{S_{\tau_0}} \Phi^2 g_{ab} t^a n^b dS + 4(\tau_1 - \tau_0)\int_{\tau}\int_{S_\tau} T_{ab} t^a n^b dS d\tau.$$

Using the energy equality, see prop. (3.3)

$$\begin{aligned}\int_{S_{\tau_1}} \left(\Phi^2 g_{ab} + T_{ab}\right) t^a dS^b \ &\leq\ 2\int_{S_{\tau_0}} \left(\Phi^2 g_{ab} + T_{ab}\right) t^a dS^b \\ &\quad +4(\tau_1 - \tau_0)\int_V \left(\Phi^2 g_{ab} + T_{ab}\right) t^a dS^b d\tau.\end{aligned}$$

Putting

$$y(\tau) = \int_{\tau_0}^{\tau}\int_{S_\tau} \left(\Phi^2 g_{ab} + T_{ab}\right) t^a dS^b d\tau,$$

we get $y'(\tau) \leq 2a + 4(\tau - \tau_0)y(\tau)$, where

$$a = \int_{S_{\tau_0}} \left(\Phi^2 g_{ab} + T_{ab}\right) t^a dS^b.$$

Note that $y(\tau_0) = 0$.

Using the above inequality and the fact that $\tau - \tau_0 \leq K$, where K is some positive constant, we can obtain an upper bound for $y(\tau)$ as follows,

$$\begin{aligned} & y'(\tau) \leq 2a + Ky(\tau) \\ \Rightarrow\ & \frac{d}{d\tau}\left[e^{-K\tau}y(\tau)\right] \leq 2ae^{-K\tau} \\ \Rightarrow\ & \int_{\tau_0}^{\tau_1} \frac{d}{d\tau}\left[e^{-K\tau}y(\tau)\right] d\tau \leq \int_{\tau_0}^{\tau_1} 2ae^{-K\tau} d\tau \\ \Rightarrow\ & y(\tau_1) \leq \frac{2a}{k}\left[e^{k(\tau_1 - \tau_0)} - 1\right] \\ \Rightarrow\ & y(\tau_1) \leq K_2 a. \end{aligned}$$

where K_2 is a positive constant. We see that $y(\tau_1)$ can be bounded above by a.

Now recall $||\Phi, V||_1^E$ is defined to be

$$||\Phi, V||_1^E = \int_\tau \int_{S_\tau} \left[\Phi^2 g_{ab} + T_{ab}\right] t^a dS^b d\tau, \tag{3.99}$$

and $||\Phi, S_{\tau_0}||_1^E$ is defined to be

$$||\Phi, S_{\tau_0}||_1^E = \int_{S_{\tau_0}} \left[\Phi^2 g_{ab} + T_{ab}\right] dS^b. \tag{3.100}$$

So we have $||\Phi, V||_1^E \leq K_2 ||\Phi, S_{\tau_0}||_1^E$ where K_2 is a positive constant. In particular if $||\Phi, S_{\tau_0}||_1^E$ is finite then

$$\int_V \Phi^2 g_{ab} t^a n^b dV < \infty,$$

since $g_{ab} t^a n^b \geq 1$ we get

$$\int_V \Phi^2 dV < \int_V \Phi^2 g_{ab} t^a n^b dV < \infty. \tag{3.101}$$

□

So in the above we have seen that we bound square integrals of Φ by $||\Phi||_{S_{\tau_0}}^{E_1}$. Using the fact that all partial derivatives of Φ are solutions to the wave equation we can repeat the above calculation and find bounds on the square integrals of the derivatives of Φ. If we go to high enough derivatives we can the apply the Sobolev embedding theorem to show the pointwise convergence of Φ given that certain norms for the initial data are finite. To proceed further we need to construct "energies" for the derivatives of Φ. As we want to construct function spaces for the initial data (the values of Φ and its normal derivative on S_{τ_0}) we will need to convert all the t and x derivatives in terms of normal and tangential derivatives to the surface S_{τ_0}. We can use the wave equation to express second order (and higher) normal derivatives in term of tangential derivatives.

The unit normal derivative to the surface S_{τ_0} is given by

$$\frac{\partial}{\partial n} = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}, \tag{3.102}$$

and the tangential derivative (orthogonal to the above normal) is given by

$$\frac{\partial}{\partial \tau} = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t}. \tag{3.103}$$

As $\mathbf{n}$ is a unit vector we have $\alpha^2 - \beta^2 = 1$ and so we can deduce that

$$\frac{\partial}{\partial t} = \alpha\frac{\partial}{\partial n} - \beta\frac{\partial}{\partial \tau}, \tag{3.104}$$

$$\frac{\partial}{\partial x} = \alpha\frac{\partial}{\partial \tau} - \beta\frac{\partial}{\partial n}. \tag{3.105}$$

Also we know that $\beta\beta,_a = \alpha\alpha,_a$ where a can be one of t, x, y, z, $\mathbf{n}$ or τ. For reasons that will become apparent later we need to express all second order and higher derivatives with respect to $\mathbf{n}$ in term of derivatives with respect to τ. Now

$$\begin{aligned} \Phi,_{\mathbf{n}\tau} &= \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t}\right)\left(\alpha\frac{\partial\Phi}{\partial t} + \beta\frac{\partial\Phi}{\partial x}\right) \\ &= (\alpha^2+\beta^2)\Phi,_{tx} + \alpha\beta(\Phi,_{tt} + \Phi,_{xx}) + \alpha,_\tau \Phi,_t + \beta,_\tau \Phi,_x \end{aligned} \tag{3.106}$$

$$\begin{aligned} \Phi,_{\tau\mathbf{n}} &= \left(\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}\right)\left(\alpha\frac{\partial\Phi}{\partial x} + \beta\frac{\partial\Phi}{\partial t}\right) \\ &= (\alpha^2+\beta^2)\Phi_{tx} + \alpha\beta(\Phi,_{tt} + \Phi,_{xx}) + \alpha,_{\mathbf{n}} \Phi,_x + \beta,_{\mathbf{n}} \Phi,_t \end{aligned} \tag{3.107}$$

$$\begin{aligned} \Phi,_{\mathbf{nn}} &= \left(\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}\right)\left(\alpha\frac{\partial\Phi}{\partial t} + \beta\frac{\partial\Phi}{\partial x}\right) \\ &= \alpha^2\Phi_{tt} + \beta^2\Phi_{xx} + 2\alpha\beta\Phi_{tx} + \alpha,_{\mathbf{n}} \Phi,_t + \beta,_{\mathbf{n}} \Phi,_x \end{aligned} \tag{3.108}$$

$$\begin{aligned}\Phi_{,\tau\tau} &= \left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial t}\right)\left(\alpha\frac{\partial\Phi}{\partial x}+\beta\frac{\partial\Phi}{\partial t}\right)\\ &= \beta^2\Phi_{,tt}+\alpha^2\Phi_{xx}+2\alpha\beta\Phi_{tx}\beta_{,\mathbf{n}}\Phi_{,t}+\alpha_{,\mathbf{n}}\Phi_{,x}\,. \qquad (3.109)\end{aligned}$$

So

$$\begin{aligned}\Phi_{,\mathbf{n}\tau}-\Phi_{,\tau\mathbf{n}} &= \alpha_{,\tau}\Phi_{,t}+\beta_{,\tau}\Phi_{,x}-\alpha_{,\mathbf{n}}\Phi_{,x}-\beta_{,\mathbf{n}}\Phi_{,t}\\ &= \alpha_{,\tau}(\alpha\Phi_{,\mathbf{n}}-\beta\Phi_{,\tau})+\beta_{\tau}(\alpha\Phi_{,\tau}-\beta\Phi_{,\mathbf{n}})-\alpha_{,\mathbf{n}}(\alpha\Phi_{,\tau}-\beta\Phi_{,\mathbf{n}})\\ &\quad -\beta_{,\mathbf{n}}(\alpha\Phi_{,\mathbf{n}}-\beta\Phi_{,\tau})\\ &= (\alpha\beta_{,\tau}-\beta\alpha_{,\tau})\Phi_{,\tau}-(\alpha\beta_{,\mathbf{n}}-\beta\alpha_{,\mathbf{n}})\Phi_{,\mathbf{n}}\,, \qquad (3.110)\\ \Phi_{,\mathbf{nn}}-\Phi_{,\tau\tau} &= \Phi_{yy}+(\alpha\beta_{,\mathbf{n}}-\beta\alpha_{,\mathbf{n}})\Phi_{\tau}-(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})\Phi_{,\mathbf{n}}\,. \qquad (3.111)\end{aligned}$$

In the (3.111) we used the wave equation to get $\Phi_{,tt}-\Phi_{,xx}=\Phi_{,yy}$.

Lemma 3.7

$$\frac{\partial^m\Phi}{\partial x^m} = \alpha_m X_m-\beta_m Y_m, \qquad (3.112)$$

$$\frac{\partial^m\Phi}{\partial t x^{m-1}} = \alpha_m Y_m-\beta_m X_m. \qquad (3.113)$$

Where $\alpha_1=\alpha,\ \beta_1=\beta,\ \beta_m=\alpha\beta_{m-1}+\beta\alpha_{m-1},\ \alpha_m=\alpha\alpha_{m-1}+\beta\beta_{m-1},\ X_1=\Phi_{,\tau}\,,\ Y_1=\Phi_{,\mathbf{n}}$ *and*

$$X_{m+1} = \frac{\partial X_m}{\partial\tau}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})Y_m-\beta\beta_m\frac{\partial^m\Phi}{\partial x^{m-2}y^2} \qquad (3.114)$$

$$Y_{m+1} = \frac{\partial Y_m}{\partial\tau}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})X_m-\beta\alpha_m\frac{\partial^m\Phi}{\partial x^{m-2}y^2}. \qquad (3.115)$$

Proof 4.1.4 The proof is by induction. Consider the case $m=2$.

$$\Phi_{,xx} = \left(\alpha\frac{\partial}{\partial\tau}-\beta\frac{\partial}{\partial\mathbf{n}}\right)\left(\alpha\frac{\partial\Phi}{\partial\tau}-\beta\frac{\partial\Phi}{\partial\mathbf{n}}\right)$$

$$
\begin{aligned}
&= \alpha^2\Phi,_{\tau\tau} + \beta^2\Phi,_{\mathbf{nn}} - \alpha\beta(\Phi,_{\mathbf{n}\tau} + \Phi,_{\tau\mathbf{n}}) \\
&\quad + \alpha\alpha,_\tau \Phi,_\tau - \beta\alpha,_{\mathbf{n}} \Phi,_\tau - \alpha\beta,_\tau \Phi,_{\mathbf{n}} + \beta\beta,_{\mathbf{n}} \Phi,_{\mathbf{n}} \\
&= \alpha^2\Phi,_{\tau\tau} + \beta^2\Phi_{\mathbf{nn}} - \alpha\beta(\Phi,_{\mathbf{n}\tau} + \Phi,_{\tau\mathbf{n}}) \\
&\quad + \alpha\beta(\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_\tau - \alpha^2(\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_{\mathbf{n}} \\
&\quad - \beta^2(\alpha\beta,_{\mathbf{n}} - \beta\alpha,_{\mathbf{n}})\Phi,_\tau + \alpha\beta(\alpha\beta,_{\mathbf{n}} - \beta\alpha,_{\mathbf{n}})\Phi,_{\mathbf{n}} \\
&= (\alpha^2 + \beta^2)(\Phi,_{\tau\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_{\mathbf{n}}) \\
&\quad - 2\alpha\beta(\Phi,_{\mathbf{n}\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_\tau) + \alpha^2\Phi,_{yy} \\
&= \alpha_2(\Phi,_{\tau\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_{\mathbf{n}} - \beta^2\Phi,_{yy} \\
&\quad - \beta_2(\Phi,_{\mathbf{n}\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi_\tau - \alpha\beta\Phi_{yy}).
\end{aligned}
$$

In a similar fashion we to get

$$
\begin{aligned}
\Phi,_{tx} &= (\alpha^2 + \beta^2)(\Phi,_{\mathbf{n}\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_\tau) \\
&\quad - 2\alpha\beta(\Phi,_{\tau\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_{\mathbf{n}}) + \alpha\beta\Phi,_{yy} \\
&= \alpha_2(\Phi_{\mathbf{n}\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_\tau - \alpha\beta\Phi,_{yy}) \\
&\quad - \beta_2(\Phi_{\tau\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_{\mathbf{n}} - \beta^2\Phi,_{yy}).
\end{aligned}
$$

So our theorem holds for $m = 2$, and now we must prove that if it holds for m then it holds for $m + 1$. Now

$$
\begin{aligned}
\alpha_{m+1}X_{m+1} - \beta_{m+1}Y_{m+1} &= \alpha_{m+1}\left(\frac{\partial X_m}{\partial \tau} - m(\alpha\beta,_\tau - \beta\alpha,_\tau)Y_m - \beta\beta_m\frac{\partial^{m+1}\Phi}{\partial x^{m-1}y^2}\right) \\
&\quad - \beta_{m+1}\left(\frac{\partial Y_m}{\partial \tau} - m(\alpha\beta,_\tau - \beta\alpha,_\tau) - \beta\alpha_m\frac{\partial^{m+1}\Phi}{\partial x^{m-1}y^2}\right) \\
&= (\alpha\alpha_m + \beta\beta_m)\left(\frac{\partial X_m}{\partial \tau} - m(\alpha\beta,_\tau - \beta\alpha,_\tau)Y_m\right)
\end{aligned}
$$

$$
\begin{aligned}
& -(\alpha\beta_m+\beta\alpha_m)\left(\frac{\partial Y_m}{\partial\tau}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})X_m\right)\\
& +\beta^2\frac{\partial^{m+1}\Phi}{\partial x^{m-1}y^2}\\
= \; & \alpha\frac{\partial}{\partial\tau}(\alpha_m X_m-\beta_m Y_m)+\beta\frac{\partial}{\partial\tau}(\beta_m X_m-\alpha_m Y_m)\\
& +\beta^2\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\beta_{m+1}X_{m+1}-\alpha_{m+1}Y_{m+1} \;=\; & \alpha\frac{\partial}{\partial\tau}(\beta_m X_m-\alpha_m Y_m)+\beta\frac{\partial}{\partial\tau}(\alpha_m X_m-\beta_m Y_m)\\
& +\alpha\beta\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right).
\end{aligned}
$$

In the above we used the fact that

$$
\alpha_m\frac{\partial\beta_m}{\partial\tau}-\beta_m\frac{\partial\alpha_m}{\partial\tau}=m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau}). \tag{3.116}
$$

This can be easily proven by expanding out α_m and β_m in terms of α_{m-1} etc. So

$$
\begin{aligned}
\frac{\partial^{m+1}\Phi}{\partial x^{m+1}} \;=\; & \left(\alpha\frac{\partial}{\partial\tau}-\beta\frac{\partial}{\partial\mathbf{n}}\right)(\alpha_m X_m-\beta_m Y_m)\\
= \; & \alpha\alpha_m\left(\frac{\partial X_m}{\partial\tau}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})Y_m\right)\\
& +\beta\beta_m\left(\frac{\partial Y_m}{\partial n}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})X_m\right)\\
& -\alpha\beta_m\left(\frac{\partial Y_m}{\partial\tau}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})X_m\right)\\
& -\beta\alpha_m\left(\frac{\partial Y_m}{\partial\mathbf{n}}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})Y_m\right)\\
= \; & \alpha\alpha_m\left(\frac{\partial X_m}{\partial\tau}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})Y_m-\beta\beta_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)\\
& -\alpha\beta_m\left(\frac{\partial Y_m}{\partial\tau}-m(\alpha\beta_{,\tau}-\beta\alpha_{,\tau})X_m-\beta\alpha_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)
\end{aligned}
$$

$$-\beta\beta_m\left(\frac{\partial Y_m}{\partial \mathbf{n}} - m(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})X_m - \alpha\alpha_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)$$
$$-\beta\alpha_m\left(\frac{\partial X_m}{\partial \mathbf{n}} - m(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})Y_m - \alpha\beta_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)$$

This must equal $\alpha_{m+1}X_{m+1} - \beta_{m+1}Y_{m+1}$. We, therefore, require that

$$\begin{aligned}
&\beta\beta_m\left(\frac{\partial Y_m}{\partial \mathbf{n}} - m(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})X_m - \alpha\alpha_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)\\
&-\beta\alpha_m\left(\frac{\partial X_m}{\partial \mathbf{n}} - m(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})Y_m - \alpha\beta_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)\\
&= \beta\beta_m\left(\frac{\partial X_m}{\partial \tau} - m(\alpha\beta_{,\tau} - \beta\alpha_{,\tau})Y_m - \beta\beta_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)\\
&-\beta\alpha_m\left(\frac{\partial Y_m}{\partial \tau} - m(\alpha\beta_{,\tau} - \beta\alpha_{,\tau})X_m - \beta\alpha_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)
\end{aligned}$$

Or, simplifying

$$\frac{\partial}{\partial \mathbf{n}}\left(\beta_m Y_m - \alpha_m X_m\right) = \frac{\partial}{\partial \tau}\left(\beta_m X_m - \alpha_m Y_m\right) + \beta\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right). \qquad (3.117)$$

Similarly

$$\begin{aligned}
\frac{\partial^{m+1}\Phi}{\partial t\partial x^m} &= \alpha\alpha_m\left(\frac{\partial Y_m}{\partial \tau} - m(\alpha\beta_{,\tau} - \beta\alpha_{,\tau})X_m - \beta\alpha_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)\\
&- \alpha\beta_m\left(\frac{\partial X_m}{\partial \tau} - m(\alpha\beta_{,\tau} - \beta\alpha_{,\tau})Y_m - \beta\beta_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)\\
&+ \beta\beta_m\left(\frac{\partial X_m}{\partial \mathbf{n}} - m(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})Y_m - \alpha\beta_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right)\\
&- \beta\alpha_m\left(\frac{\partial Y_m}{\partial \mathbf{n}} - m(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})X_m - \alpha\alpha_m\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right).
\end{aligned}$$

This must equal $\alpha_{m+1}Y_{m+1} - \beta_{m+1}X_{m+1}$. After a similar calculation to the one above this implies that

$$\frac{\partial}{\partial \mathbf{n}}\left(\beta_m X_m - \alpha_m Y_m\right) = \frac{\partial}{\partial \tau}\left(\beta_m Y_m - \alpha_m X_m\right) - \alpha\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right). \qquad (3.118)$$

So to prove our lemma it remains for us to prove that the two equations below hold.

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{n}}(\beta_m Y_m - \alpha_m X_m) &= \frac{\partial}{\partial \tau}(\beta_m X_m - \alpha_m Y_m) + \beta\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right), \\
\frac{\partial}{\partial \mathbf{n}}(\beta_m X_m - \alpha_m Y_m) &= \frac{\partial}{\partial \tau}(\beta_m Y_m - \alpha_m X_m) - \alpha\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right).
\end{aligned}$$

We will prove this, as previously stated, by induction. The above are both true for $m = 1$. We assume the m^{th} case is true and prove $m + 1$. Now

$$\begin{aligned}
&\frac{\partial}{\partial \tau}(\alpha_{m+1}X_{m+1} - \beta_{m+1}Y_{m+1}) \\
= \; & \frac{\partial}{\partial \tau}\Bigg(\alpha\frac{\partial}{\partial \tau}(\alpha_m X_m - \beta_m Y_m) + \beta\frac{\partial}{\partial \tau}(\beta_m X_m - \alpha_m Y_m) \\
&+\beta^2\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\Bigg) \\
= \; & \frac{\partial}{\partial \tau}\Bigg(\alpha\frac{\partial}{\partial n}(\alpha_m Y_m - \beta_m X_m) + \beta\frac{\partial}{\partial n}(\beta_m Y_m - \alpha_m X_m) \\
&-\alpha^2\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\Bigg) \\
= \; & \frac{\partial}{\partial \tau}\left(-\alpha^2\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right) \\
&+\alpha\frac{\partial^2}{\partial n\partial \tau}(\alpha_m Y_m - \beta_m X_m) + \beta\frac{\partial^2}{\partial n\partial \tau}(\beta_m Y_m - \alpha_m X_m) \\
&+\alpha(\alpha\beta_{,\tau} - \beta\alpha_{,\tau})\frac{\partial}{\partial \tau}(\alpha_m Y_m - \beta_m X_m) - \alpha(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})\frac{\partial}{\partial n}(\alpha_m Y_m - \beta_m X_m) \\
&+\beta(\alpha\beta_{,\tau} - \beta\alpha_{,\tau})\frac{\partial}{\partial \tau}(\beta_m Y_m - \alpha_m X_m) - \beta(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})\frac{\partial}{\partial n}(\beta_m Y_m - \alpha_m X_m) \\
&+\beta(\alpha\beta_{,\tau} - \beta\alpha_{,\tau})\frac{\partial}{\partial n}(\alpha_m Y_m - \beta_m X_m) + \alpha(\alpha\beta_{,\tau} - \beta\alpha_{,\tau})(\beta_m Y_m - \alpha_m X_m) \\
= \; & \frac{\partial}{\partial \tau}\left(-\alpha^2\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right) + \alpha\frac{\partial^2}{\partial n\partial \tau}(\alpha_m Y_m - \beta_m X_m) \\
&+\beta\frac{\partial^2}{\partial n\partial \tau}(\beta_m Y_m - \alpha_m X_m) + \alpha(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})(\beta_m Y_m - \alpha_m X_m) \\
&+\Big(\alpha\beta(\alpha\beta_{,\tau} - \beta\alpha_{,\tau}) - \alpha^2(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})\Big)\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right) \\
&-\beta(\alpha\beta_{,\mathbf{n}} - \beta\alpha_{,\mathbf{n}})(\beta_m X_m - \alpha_m Y_m)
\end{aligned}$$

$$
\begin{aligned}
& + \left(\alpha\beta(\alpha\beta,_\tau - \beta\alpha,_\tau) - \beta^2(\alpha\beta,_\mathbf{n} - \beta\alpha,_\mathbf{n})\right)\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right) \\
= \; & \frac{\partial}{\partial n}\left[\alpha\frac{\partial}{\partial\tau}\left(\alpha_m Y_m - \beta_m X_m\right) + \beta\frac{\partial}{\partial\tau}\left(\beta_m Y_m - \alpha_m X_m\right)\right] \\
& -\alpha_2(\alpha\beta,_\mathbf{n} - \beta\alpha,_\mathbf{n})\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right) \\
& -\beta_2(\alpha\beta,_\tau - \beta\alpha,_\tau)\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right) \\
& -\alpha^2\left(\frac{\partial^{m+2}\Phi}{\partial x^{m-1}\partial\tau\partial y^2}\right) \\
& +\beta_2(\alpha\beta,_\tau - \beta\alpha,_\tau)\left(\frac{\partial^{m+2}\Phi}{\partial x^m\partial y^2}\right) \\
= \; & \frac{\partial}{\partial n}\left[\alpha\frac{\partial}{\partial\tau}\left(\alpha_m Y_m - \beta_m X_m\right) + \beta\frac{\partial}{\partial\tau}\left(\beta_m Y_m - \alpha_m X_m\right)\right] \\
& -\left(\alpha_2(\alpha\beta,_\mathbf{n} - \beta\alpha,_\mathbf{n}) + \beta_2(\alpha\beta,_\tau - \beta\alpha,_\tau)\right)\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right) \\
& -\alpha\beta\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right) - \alpha\left(\frac{\partial^{m+2}\Phi}{\partial x^m\partial y^2}\right) \\
& +\beta_2(\alpha\beta,_\tau - \beta\alpha,_\tau)\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right) \\
= \; & \frac{\partial}{\partial n}\left[\alpha\frac{\partial}{\partial\tau}\left(\alpha_m Y_m - \beta_m X_m\right) + \beta\frac{\partial}{\partial\tau}\left(\beta_m Y_m - \alpha_m X_m\right)\right. \\
& \left. -\alpha\beta\left(\frac{\partial^{m+1}\Phi}{\partial x^{m-1}\partial y^2}\right)\right] - \alpha\left(\frac{\partial^{m+2}\Phi}{\partial x^m\partial y^2}\right) \\
= \; & \frac{\partial}{\partial n}\left(\alpha_{m+1}Y_{m+1}\beta_{m+1}X_{m+1}\right) - \alpha\left(\frac{\partial^{m+2}\Phi}{\partial x^m\partial y^2}\right),
\end{aligned}
$$

as required. The second equation can be proven in an identical manner

□

We can now use the above expressions for the m^{th} x derivative of Φ and $\Phi_{,t}$ to construct our higher order norms. Now the first order energy norm on the hypersurface

S_{τ_0} is given by:

$$\int_{S_{\tau_0}} \alpha \left[\Phi^2 + \Phi^2,_t + \Phi^2,_x + \Phi^2,_y\right] + 2\beta\Phi,_t \Phi,_x \, dS. \tag{3.119}$$

We re-write this in terms of the normal and tangential derivatives and we get:

$$\int_{S_{\tau_0}} \alpha \left[\Phi^2 + (\alpha\Phi,_{\mathbf{n}} - \beta\Phi_\tau)^2 + (\alpha\Phi,_\tau - \beta\Phi,_{\mathbf{n}})^2 + \Phi^2,_y\right]$$
$$+2\beta(\alpha\Phi,_{\mathbf{n}} - \beta\Phi,_\tau)(\alpha\Phi,_\tau - \beta\Phi,_{\mathbf{n}}) dS.$$

Simplifying this we get

$$\int_{S_{\tau_0}} \alpha \left[\Phi^2 + \Phi^2,_{\mathbf{n}} + \Phi^2,_\tau + \Phi^2,_y\right] - 2\beta\Phi,_{\mathbf{n}} \Phi,_\tau \, dS. \tag{3.120}$$

As $\alpha^2 - \beta^2 = 1$ we have $\alpha > |\beta|$ we we can bound the above by

$$K_3 \int_{S_{\tau_0}} \alpha \left[\Phi^2 + \Phi^2,_{\mathbf{n}} + \Phi^2,_\tau + \Phi^2,_y\right] dS, \tag{3.121}$$

where K_3 is a positive constant. We now construct a second order energy norm. This is obtained by adding together the first order energy norms for the first order t, x and y derivatives of Φ. We get

$$\begin{aligned} \int_{S_{\tau_0}} \alpha & \left[\Phi^2,_t + \Phi^2,_x + \Phi^2,_y + \Phi^2,_{tt} + \Phi^2,_{xx} + \Phi^2,_{yy} + 2\Phi^2,_{tx} + 2\Phi^2,_{ty}\right. \\ + & \left. 2\Phi,_{xy}\right] \\ + & 2\beta\left[\Phi,_{tt}\Phi,_{tx} + \Phi,_{tx}\Phi,_{xx} + \Phi,_{ty}\Phi,_{xy} + \Phi,_{tz}\Phi,_{xz}\right] dS. \end{aligned}$$

Now using the wave equation we eliminate all second order t derivatives, giving

$$\int_{S_{\tau_0}} \alpha \left[\Phi^2,_t + \Phi^2,_x + \Phi^2,_y + \Phi,^2_{yy} + 2\Phi,_{xx}\Phi,_{yy}\right.$$
$$\left. + 2\Phi^2,_{xx} + \Phi^2,_{yy} + 2\Phi^2,_{tx} + 2\Phi^2,_{ty} + 2\Phi,_{xy}\right]$$
$$+2\beta\left[\Phi,_{yy}\Phi,_{tx} + 2\Phi,_{tx}\Phi,_{xx} + \Phi,_{ty}\Phi,_{xy}\right] dS.$$

Converting to normal and tangential derivatives we get

$$\int_{S_{\tau_0}} \alpha \Big[(\alpha\Phi,_{\mathbf{n}} - \beta\Phi,_\tau)^2 + (\alpha\Phi,_\tau - \beta\Phi,_{\mathbf{n}})^2 + \Phi^2,_y + 2(\alpha_2 X_2 - \beta_2 Y_2)^2$$
$$+2(\alpha_2 Y_2 - \beta_2 X_2)^2 + 2(\alpha_2 X_2 - \beta_2 Y_2)\Phi,_{yy} + 2\Phi,^2_{yy}$$
$$+2(\alpha\Phi,_{\mathbf{n}y} - \beta\Phi,_{\tau y})^2 + 2(\alpha\Phi,_{\tau y} - \beta\Phi,_{\mathbf{n}y})^2\Big]$$
$$+2\beta\left[2(\alpha_2 X_2 - \beta_2 Y_2)(\alpha_2 Y_2 - \beta_2 X_2) + (\alpha_2 Y_2 - \beta_2 X_2)\Phi,_{yy}\right.$$
$$\left.+(\alpha\Phi_{\mathbf{n}y} - \beta\Phi,_{\tau y})(\alpha\Phi,_{\tau y} - \beta\Phi,_{\mathbf{n}y})\right] dS$$
$$= \int_{S_{\tau_0}} \alpha\Big[\alpha_2(\Phi^2,_{\mathbf{n}} + \Phi^2,_\tau) + \Phi^2,_y + 2\Phi^2_{yy}$$
$$+\alpha_2(\Phi^2,_{\mathbf{n}y} + \Phi^2,_{\tau y}) - 2\beta_2\Phi,_{\mathbf{n}y}\Phi,_{\tau y}\Big]$$
$$\alpha\Big[\Phi^2,_{\mathbf{n}y} + \Phi^2,_{\tau y}\Big] - 2\beta\left[\Phi,_{\mathbf{n}y}\Phi,_{\tau y}\right]$$
$$+2\alpha_3(X_2^2 + Y_2^2) - 4\beta_3 X_2 Y_2 + (\alpha X_2 - \beta Y_2)\Phi,_{yy}\, dS.$$

In the above

$$X_2 = \Phi,_{\tau\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_{\mathbf{n}} - \beta^2\Phi,_{yy}\,, \tag{3.122}$$

$$Y_2 = \Phi,_{\mathbf{n}\tau} - (\alpha\beta,_\tau - \beta\alpha,_\tau)\Phi,_\tau - \alpha\beta\Phi,_{yy}\,, \tag{3.123}$$

Now for our Cauchy surface $x^2 - t^2 = C^2$ the quantity $(\alpha\beta,_\tau - \beta\alpha,_\tau)$ is a constant $(= C^{-1})$ and so the second order energy norm can be bound above by

$$K_3 \int_{S_{\tau_0}} \alpha^3(\Phi^2,_{\mathbf{n}\tau} + \Phi^2,_{\tau\tau} + \Phi^2,_{\mathbf{n}y} + \Phi^2,_{\tau y} + \Phi^2,_{yy} + \Phi^2,_{\mathbf{n}} + \Phi^2,_\tau + \Phi^2,_y)dS, \tag{3.124}$$

where K_3 is a positive constant. Following a similar calculation to the above the third order energy norm can be bounded above by

$$K_4 \int_{S_{\tau_0}} \alpha^5(\Phi^2,_{\mathbf{n}\tau\tau} + \Phi^2,_{\tau\tau\tau} + \Phi^2,_{\tau yy} + \Phi^2,_{\mathbf{n}yy} + \Phi^2,_{\mathbf{n}\tau y}$$

$$+\Phi^2{}_{,\tau\tau y} + \Phi^2{}_{,\mathbf{n}yz} + \Phi^2{}_{,\tau yz} + \Phi^2{}_{,yyy} + \Phi^2{}_{,\mathbf{n}\tau} + \Phi^2{}_{,\tau\tau} + \Phi^2{}_{,yy} + \Phi^2{}_{,\mathbf{n}y} \quad (3.125)$$

$$+\Phi^2{}_{,\tau y} + \Phi^2{}_{,\tau} + \Phi^2{}_{,\mathbf{n}})dS,$$

where K_4 is a positive constant. We can similar find bounds for higher order energy norms. Now let $E_m(\Phi)$ denote the m^{th} order energy norm of Φ. If $E(\Phi)$ is the energy for the scalar field on S_{τ_0} of Φ then $E_m(\Phi)$ is given by

$$E_m(\Phi) = \int_{S_{\tau_0}} \alpha |\partial^m \Phi|^2 dS + E(\partial^m \Phi), \quad (3.126)$$

where derivatives are taken in the t, x, and y directions. Generalising the calculations done above for the second and third order norms we see that the m^{th} order energy norm can be bounded above by

$$K_m \int_{S_{\tau_0}} \alpha^{2m-1} (\sum_{i=1}^{m} |D^i \Phi|^2 + \sum_{i=0}^{m-1} |D^i \Phi_{,\mathbf{n}}|^2) dS, \quad (3.127)$$

where K_m is a positive constant and derivatives are taken in directions tangent to the surface S_{τ_0}.

Definition 3.8 *Let $W_E^m(S)$ denote the energy Sobolev space which has norm*

$$\begin{aligned} ||\Phi, S_{\tau_0}||_m^E &= \int_{S_{\tau_0}} \left[\Phi^2 g_{ab} t^a n^b + \sum_{i=1}^{m} E_i(\Phi) \right] dS \quad (3.128) \\ &= \int_{S_{\tau_0}} \left[\alpha \Phi^2 + \sum_{i=1}^{m} E_i(\Phi) \right] dS, \end{aligned}$$

on the surface $\tau = \tau_0$ and with spacetime norm

$$\begin{aligned} ||\Phi, V||_m^E &= \int_\tau \int_{S_\tau} \left[\Phi^2 g_{ab} t^a n^b + \sum_{i=1}^{m} E_i(\Phi) \right] dS d\tau \quad (3.129) \\ &= \int_\tau \int_{S_\tau} \left[\alpha \Phi^2 + \sum_{i=1}^{m} E_i(\Phi) \right] dS d\tau. \end{aligned}$$

Now we saw above that $E_m(\Phi)$ could be bounded above by integrals of the form

$$K_m \int_{S_{\tau_0}} \alpha^{2m-1}(\sum_{i=1}^{m} |D^i\Phi|^2 + \sum_{i=0}^{m-1} |D^i\Phi,_{\mathbf{n}}|^2)dS. \tag{3.130}$$

Converting from t and x derivatives to normal and tangential derivatives results in us picking up factors of α in our norms. As our initial data surface S_{τ_0} is not compact, these terms cannot be bounded by constants. As a result we will need to form some kind of weighted Sobolev space.

Definition 3.9 *Define the weighted energy Sobolev spaces $W^m_{E,\alpha}$ to be function spaces with norms given by $||\Phi, S_{\tau_0}, \alpha||^E_m$ where*

$$||\Phi, S_{\tau_0}, \alpha||^E_m = \int_{S_{\tau_0}} \alpha|\Phi|^2 + \alpha^{2m-1} \sum_{i=1}^{m} |D^i\Phi| dS, \tag{3.131}$$

where derivatives are taken in directions tangential to the surface S_{τ_0}.

We have now formed all the function spaces we need to complete an existence proof for solutions to the wave equation in Minkowski space given initial data on the spacelike hypersurface $x^2 - t^2 = C^2$. In what follows S_{τ_0} is the surface $x^2 - t^2 = C^2$, and V is $D^+(S_{\tau_0}) \cup H^+(S_{\tau_0})$. V is foliated by surfaces $\tau = constant.$ In the following all the C_i are positive constants.

Proposition 3.10

$$||\Phi, V||^m \leq C_3 ||\Phi, S_{\tau_0}||^E_{m+1}. \tag{3.132}$$

Proof

Recall that

$$\|\Phi, V\|^0 = \int_V \Phi^2 dV \leq C_1\|\Phi, V\|_1^E \leq C_2\|\Phi, S_{\tau_0}\|_1^E.$$

We can duplicate the above result with Φ replaced by its partial derivative with respect to t, x, y and z as they to are solutions to the wave equation. If we then add together these inequalities for each derivative of Φ we then get

$$\|\Phi, V\|^m \leq C_3\|\Phi, S_{\tau_0}\|_{m+1}^E. \tag{3.133}$$

□

Lemma 3.11

$$\|\Phi, S_{\tau_0}\|_{m+1}^E \leq C_4(\|\Phi, S_{\tau_0}, \alpha\|_{m+1}^E + \|\Phi_{,\mathbf{n}}, S_{\tau_0}, \alpha\|_m^E). \tag{3.134}$$

Proof

We have see that $E_m(\Phi)$ is bounded above by terms like

$$K_m \int_{S_{\tau_0}} \alpha^{2m-1}(\sum_{i=1}^{m} |D^i\Phi|^2 + \sum_{i=0}^{m-1} |D^i\Phi_{,\mathbf{n}}|^2)dS. \tag{3.135}$$

Summing this result for all m and separating out the normal derivatives we get our result.

□

Therefore:

$$\|\Phi, V\|^m \leq C_5(\|\Phi, S_{\tau_0}, \alpha\|^E_{m+1} + \|\Phi_{,\mathbf{n}}, S_{\tau_0}, \alpha\|^E_m). \tag{3.136}$$

Using the Sobolev embedding theorem we see that for $m = 4 + p$, where $p \geq 0$

$$|D^p\Phi| < C_p(\|\Phi, S_{\tau_0}, \alpha\|^E_{5+p} + \|\Phi_{,\mathbf{n}}, S_{\tau_0}, \alpha\|^E_{4+p}). \tag{3.137}$$

This inequality proves the continuous dependence of Φ on its initial data. That is, defining a topology on the space of solutions via the weighted energy Sobolev space norm, we see that a linear map from the initial data to the solutions is bounded and hence continuous.

Before we prove our main theorem below we will first give some definitions.

Definition 3.12 *Consider a complete normed vector space X. The sequence (f_n) in X converges strongly to $f \in X$ if*

$$\lim_{n\to\infty} \|f - f_n\| = 0, \tag{3.138}$$

which implies

$$\lim_{n\to\infty} \|f_n\| = \|f\|. \tag{3.139}$$

Definition 3.13 *Let X be a topological vector space and let X' be its topological dual. The sequence (f_n) in X converges weakly to $f \in X$ if for every $g \in X'$*

$$< g, f_n > \to < g, f >, \tag{3.140}$$

as $n \to \infty$. Weak convergence in a normed space implies

$$\|f\| \leq \lim_{n\to\infty} \inf \|f_n\|. \tag{3.141}$$

Theorem 3.14 *Let our initial data, Φ_0 and Φ_1, be such that*

$$\Phi_0 \in W^E_{m+1,\alpha}(S_{\tau_0}),\ \Phi_1 \in W^E_{m,\alpha}(S_{\tau_0}), \tag{3.142}$$

then there exists a unique solution $\Phi \in W^m(V)$ to the wave equation.

Proof

We will prove this by approximating the initial data by analytic fields and showing that the analytic solutions obtained converge to a scalar field which is a solution to the wave equation with the given initial conditions. Let Φ_{0_n} and Φ_{1_n} be analytic fields on S_{τ_0} which converge strongly to Φ_0 and Φ_1 in $W^E_{m+1,\alpha}(S_{\tau_0})$ and $W^E_{m,\alpha}(S_{\tau_0})$ respectively. For each n there is an analytic solution, Φ_n, to the wave equation with initial data Φ_{0_n} and Φ_{1_n}. By lemma (3.6) $\|\Phi_n, V\|^m$ is bounded above as $n \to \infty$. Thus $\|D^b\Phi_n, V\|^0$ is bounded above for all $b \leq m$. Given that $W^0(V)$ is a Banach space we can use the following result [CDD]:

Theorem: From a bounded sequence in a Banach space, one can always extract a subsequence which converges weakly

(This is the functional analysis version of *every bounded sequence has a convergent subsequence.*)

So, using the above, there will be a field $\Phi \in W^m(V)$ and a subsequence $\Phi_{n'}$ of Φ_n such that for each $b \leq m$, $D^b\Phi_{n'}$ converges weakly to $D^b\Phi_n$.

As a result of the above, $\Box\Phi_n$ will converge weakly to $\Box\Phi$. But $\Box\Phi_n = 0$, therefore

$\Box\Phi = 0$. We also have that Φ_{0_n} and Φ_{1_n} converge weakly to Φ_0 and Φ_1 on S_{τ_0}. So we see Φ is a solution to the wave equation with the given initial conditions. Since $\Phi_0 \in W^E_{m+1,\alpha}(S_{\tau_0})$ and $\Phi_1 \in W^E_{m,\alpha}(S_{\tau_0})$ then by proposition (4.1.?) $\Phi \in W^m(V)$. By proposition (3.4) this Φ will be a unique solution to the wave equation.

□

Corollary 3.15 *Let $\Phi_0 \in W^E_{m+1,\alpha}(S_{\tau_0})$ and $\Phi_1 \in W^E_{m,\alpha}(S_{\tau_0})$, where $m = 2 + p$ ($p = 0, 1, 2, \ldots$), then Φ is a C^p solution to the wave equation.*

Proof

Now for $2m > n$, where n is the dimension of the spacetime, and $\Phi_0 \in W^E_{m+1,\alpha}(S_{\tau_0})$ and $\Phi_1 \in W^E_{m,\alpha}(S_{\tau_0})$ we have $|\Phi| \leq \|\Phi, V\|^m$ in V by the Sobolev embedding theorem. Taking $n = 3$ we will have a pointwise convergent solution to the wave equation for $m \geq 2$. Taking $m = 2 + p$ we will have a C^p solution to the wave equation.

□

In particular this means we can find function spaces for the initial data such that Φ and T_{ab} (for the scalar field Φ) are C^p on the horizon (i.e. the limit of Φ, T_{ab} and their p^{th} derivatives are finite as one approaches the horizon).

3.1.3 Sobolev Spaces on Gott Space

In the above all the function spaces and their norms we have constructed are for Minkowski space. Our aim is to prove the existence of solutions to the wave equation in Gott space. We therefore need to construct function spaces for the data on the physical spacetime ($-d \leq y \leq d$), and somehow relate these to the function spaces on Minkowski space.

Let us first give a more detailed explanation of how the data was constructed in the covering space for Gott space. Let Φ_P be a function on the physical Gott space, and let Φ_C be the equivalent function on the covering space. Let the operator L represent the action of the Gott space isometries on a point $\mathbf{p}$. That is, if $\mathbf{p}$ is a point in the physical space then its equivalent image point in the covering space $\mathbf{p}_n$ is given by (see eqns (2.86) and (2.88))

$$L_n\mathbf{p} \rightarrow \mathbf{p}_n. \tag{3.143}$$

For a function Φ_C on the covering space to represent a function Φ_P on the physical space we must have

$$\Phi_C(\mathbf{p}) = \Phi_C(\mathbf{p}_n), \tag{3.144}$$

for all n. Now let Φ_P represent initial data for the Cauchy problem on the physical space. We now regard this as initial data, Φ_C, in the covering space and translate it for values of $y^2 > d^2$ so that

$$\Phi_P(\mathbf{p}) = \Phi_C(\mathbf{p}), \tag{3.145}$$

for $-d \leq y \leq y$ and

$$\Phi_C(\mathbf{p}) = \Phi_C(\mathbf{p}_n), \tag{3.146}$$

for all n, and $y^2 > d^2$.

We now have now converted our Gott space problem to an equivalent problem in Minkowski space.

In the physical space ($-d \leq y \leq d$) null geodesics spiral round both strings only a finite number of times as they work their way from the the initial data ($t = -\sqrt{(x^2 + C^2)}$ surface to the field point. The fact that geodesics wrap around a finite number of times n in the physical space given by

$$n = \left[\frac{d + C - \sqrt{T^2 - X^2} + Y}{4d} \right], \tag{3.147}$$

means that when we go to the covering space the intersection of the past light cone of the field point with the Cauchy surface only extends a finite amount in the y-direction (see the beginning of this chapter for more details) with value:

$$K = Y + C - \sqrt{T^2 - X^2}, \tag{3.148}$$

where (T, X, Y) is the location of the field point.

We now consider forming function spaces in the physical space, and relating them to the equivalent function spaces in the covering space. Let $F_C^m(S_C)$ be a function space in the covering space on some domain S_C with norm $||\Phi_C, S_C||_C^m$, where y ranges between $Y - C$ and $Y + C$ on S_C. F_C^m can be any of the function spaces constructed in the previous chapter. Denote the equivalent function space in the physical space by $F_P^m(S_P)$ with norm $||\Phi_P, S_P||_P^m$, where the y range of integration is from $-d$ to d . In the above S_C is obtained from the unwrapping of S_P in the covering space. Note that

$$||\Phi_P, S_P||_P^m = ||\Phi_C, S_C||_C^m, \tag{3.149}$$

if we restrict the y range of the integration in $||\Phi_C, S_C||_C^m$ to lie between $-d$ and d. In fact, given the periodic nature of Φ_C, we get

$$\begin{aligned} ||\Phi_C, S_C : -nd \leq y \leq nd||_C^m &= 2n||\Phi_C, S_C : -d \leq y \leq d||_C^m \\ &= 2n||\Phi_P, S_P||_P^m. \end{aligned} \tag{3.150}$$

As the y range of integration is always finite in the covering space this means there exit constants C_6 and C_7 such that

$$C_6||\Phi_C, S_C||_C^m \leq ||\Phi_P, S_P||_P^m \leq C_7||\Phi_C, S_C||_C^m. \tag{3.151}$$

Therefore the norms $||\Phi_C, S_C||_C^m$ and $||\Phi_P, S_P||_P^m$ can be taken as equivalent. That is, if $\Phi \in F_P^m$ then $\Phi \in F_C^m$.

Definition 3.16 *Define the weighted energy Sobolev spaces $W_{P,E,\alpha}^m$ on the physical space to be function spaces with norms given by $||\Phi_P, S_{\tau_0}, \alpha||_{P,m}^E$ where $||\Phi_P, S_{\tau_0}, \alpha||_{P,m}^E$ is given by*

$$\int_{S_{\tau_0}} \alpha|\Phi_P|^2 + \alpha^{2m-1}\sum_{i=1}^{m}|D^i\Phi_P|dS, \tag{3.152}$$

where derivatives are taken in directions tangential to the surface S_{τ_0} and the y integration ranges between $-d$ and d.

In a similar way define $W_P^m(V)$ to be the Sobolev space on the physical space.

Corollary 3.17 *Let Φ_{P0} and Φ_{P1} be the values of Φ_P and its normal derivative respectively on the surface S_{τ_0} in the physical space. If $\Phi_{P0} \in W^E_{P,4+p,\alpha}(S_{\tau_0})$ and $\Phi_{P1} \in W^E_{P,3+p,\alpha}(S_{\tau_0})$ then Φ_P is a C^p solution to the wave equation in Gott space. Here S_{τ_0} is the surface $t^2 - x^2 = \tau_0^2$, $-d \leq y \leq d$.*

Proof

Φ_{P0} and Φ_{P1} can be unwrapped to give data on the covering space, Φ_{C0} and Φ_{C1} respectively. If $\Phi_{P0} \in W^E_{P,4+p,\alpha}(S_{\tau_0})$ and $\Phi_{P1} \in W^E_{P,3+p,\alpha}(S_{\tau_0})$ then $\Phi_{C0} \in W^E_{4+p,\alpha}(S_{\tau_0})$ and $\Phi_{C1} \in W^E_{3+p,\alpha}(S_{\tau_0})$. By corollary (4.1.10) we have a unique solution, $\Phi_C \in W^m(V)$, to the wave equation in the covering space. If $\Phi_C \in W^m(V)$ then $\Phi_P \in W^m_P(V)$. For $2m > n$, where $n = 4$ we have a unique solution, Φ_P, to the wave equation in the physical Gott space by the Sobolev embedding theorem. In particular this means we can find function spaces for the initial data in the physical Gott space such that Φ and T_{ab} (for the scalar field Φ) are C^p on the horizon (i.e. the limit of Φ, T_{ab} and their P^{th} derivatives are finite as one approach the horizon in the chronal region).

3.2 The Fourier Approach

In the previous section the approach we took to solving the Cauchy problem only allowed us to consider solutions to the wave equation up to the chronology horizon. It appears that these solutions remain finite as we approach the horizon, but we cannot say anything about their behaviour beyond the chronology horizon. In what follows we will use a Fourier analysis approach (described below) that will allow us to look at solutions to the wave equation which go beyond the horizon.

We first consider the wave equation in Misner space using this approach. Recall the metric for Misner space is given by

$$ds^2 = 2d\xi d\tau + \tau d\xi^2, \tag{3.153}$$

where $-\infty < \tau < \infty$ and $0 \leq \xi \leq 2\pi$. The wave equation is given by

$$\Box\Phi = (-g)^{-\frac{1}{2}}\, \partial_a \left[(-g)^{\frac{1}{2}}\, g^{ab}\partial_b\Phi\right] = 0, \tag{3.154}$$

which becomes

$$\tau\partial_\tau^2\Phi + \partial_\tau\Phi - 2\partial_\xi\partial_\tau\Phi = 0. \tag{3.155}$$

The characteristics of (3.155) are given by $\xi = k$ and $\xi + 2\ln\tau = K$, where k and K are both constants. These simply correspond to the null geodesics of the spacetime. Now the solution to this equation must be periodic in ξ, with period 2π, so putting

$$\Phi(\tau, \xi) = \Re\left(\sum_{m=0}^{\infty} e^{im\xi}\phi_m(\tau)\right), \tag{3.156}$$

we get

$$\tau\partial_\tau^2\phi_m + (1 - 2im)\partial_\tau\phi_m = 0. \tag{3.157}$$

This equation has two solutions

$$\phi_m(\tau) = B_m,\ \phi_m(\tau) = C_m\tau^{2im} \tag{3.158}$$

where B_m and C_m are complex constants. (3.158) gives a solution of the full wave equation (3.155) as

$$\Phi(\tau,\xi) = \Re\left(\sum_m \left(B_m e^{im\xi} + C_m e^{im(\xi+2\ln\tau)}\right)\right), \tag{3.159}$$

where B_m and C_m are complex constants.

Now consider the stress-energy tensor T_{ab} of (3.159) for the scalar field Φ. This is given by

$$T_{ab} = \Phi_{,a}\Phi_{,b} - \frac{1}{2} g_{ab} g^{cd} \Phi_{,c}\Phi_{,d}\,. \tag{3.160}$$

So we get for the different components:

$$\begin{aligned} T_{00} &= \Phi^2{}_{,\tau} \\ &= \frac{4}{\tau^2}\left(\Re\left(\sum_m imC_m e^{im(\xi+2\ln\tau)}\right)\right)^2, && (3.161) \\ T_{11} &= \Phi^2{}_{,\xi} - \frac{1}{2} g_{\xi\xi}(-\tau\Phi^2{}_{,\tau} + 2\Phi_{,\xi}\Phi_{,\tau}) \\ &= \left(\Re\left(\sum_m iB_m e^{im\xi}\right)\right)^2 + \left(\Re\left(\sum_m iC_m e^{im(\xi+\ln\tau)}\right)\right)^2, && (3.162) \\ T_{01} &= \frac{1}{2}\tau\Phi^2{}_{,\tau} \\ &= \frac{2}{\tau}\left(\Re\left(\sum_m imC_m e^{im(\xi+2\ln\tau)}\right)\right)^2. && (3.163) \end{aligned}$$

Now consider the scalar quantity $T^{ab}T_{ab}$ given by

$$\begin{aligned} T_{ab}T^{ab} &= \frac{1}{2}(g^{ab}\Phi_{,a}\Phi_{,b})^2 \\ &= \frac{1}{2}(-\tau\Phi^2{}_{,\tau} + 2\Phi_{,\tau}\Phi_{,\xi})^2 \\ &= \frac{8}{\tau^2}\left(\Re\left(\sum_m mB_m e^{im\xi}\sum_m mC_m e^{im(\xi+2\ln\tau)}\right)\right)^2. \end{aligned}$$

So we see that T_{00} and T_{01} will diverge at the chronology horizon, $\tau = 0$, unless $C_m = 0$. The physical reason for this is that C_m is the coefficient of the part of the solution that corresponds to geodesics that wind round and round the spacetime as they approach $\tau = 0$. It is radiation travelling along these geodesics that gets infinitely blue shifted and causes the instability in Misner space. The quantity $T_{ab}T^{ab}$ is divergent unless either $B_m = 0$ for $m = 1, 2, ...$ or $C_m = 0$ for $m = 1, 2, ...$ This is consistent with the results earlier in the chapter. There we saw that if the initial value of Φ was either a function of τ or $\xi + 2\ln\tau$ then it evolved into a finite solution of the wave equation, however if the initial value is a mixture of these two it evolves to a solution whose stress-energy scalar becomes divergent as we approach the chronology horizon. This is indeed what happens in the Fourier case we have considered here. Only if the solution of the wave equation is a function of τ or $\xi + 2\ln\tau$, i.e. if B_m or C_m are equal to zero, do we get a finite stress-energy scalar.

Now we consider solving the wave equation in Grant space [G]. Recall that Grant space is the quotient of Minkowski space $(M, \mathbf{g})$ under the group of isometries G. If A is a generator of G then A^n maps a point (t, x, y) to the point

$$(t\cosh 4na + x\sinh 4na, x\cosh 4na + t\sinh 4na, y + 4nd), \qquad (3.164)$$

Now any solution to the wave equation constructed on Grant space is going to be periodic under the above boost in t, x and translation in y. One way to solve the wave equation with periodic solutions is by Fourier analysis. What we need to do is make some coordinate changes so that all the isometries of Gott space are embodied in one coordinate. We can then proceed to Fourier analyse the solution of the wave equation in terms of that coordinate.

First consider solving the problem in the globally hyperbolic region. In this region orbits of the Grant space isometries are given by hyperbolae of the form $t^2 - x^2 > 0$. Define new coordinates τ and ξ by

$$t = \tau \cosh \xi, \;\; x = \tau \sinh \xi. \tag{3.165}$$

So here τ labels the different orbits, and ξ varies within these orbits.

Under the boost

$$\begin{aligned} t = \tau \cosh \xi &\rightarrow \tau \cosh \xi \cosh 4na + \tau \sinh \xi \sinh 4na \\ &= \tau \cosh(\xi + 4na), \\ x = \tau \sinh \xi &\rightarrow \tau \sinh \xi \cosh 4na + \tau \cosh \xi \sinh 4na \\ &= \tau \sinh(\xi + 4na). \end{aligned}$$

So the action of the boost is to make $\xi \rightarrow \xi + 4na$. The other isometry is the translation in y, $y \rightarrow y + 4nd$.

Now define another set of new coordinates μ and ν by

$$\mu = \frac{a}{d} y + \xi, \;\; \nu = \frac{a}{d} y - \xi. \tag{3.166}$$

Under the Grant space isometries given by (3.164)

$$\begin{aligned} \mu &\rightarrow \frac{a}{d}(y + 4nd) + \;(\xi + 4na) \\ &= \frac{a}{d} y + \xi \end{aligned}$$

$$
\begin{aligned}
&= \mu, \\
\nu &\rightarrow \frac{a}{d}(y+4nd) - \ (\xi + 4na) \\
&= \frac{a}{d}y - \xi - 8na \\
&= \nu - 8na.
\end{aligned}
$$

So if we change coordinates from (t, x, y) to (τ, μ, ν) we find we have a coordinate system such that all the periodicities of Grant space are manifest in the coordinate ν. We can now write the wave equation in terms of these new coordinates and Fourier analyse in ν. Our new coordinates are

$$\tau = \sqrt{t^2 - x^2},\ \mu = -\frac{a}{d}y + \xi,\ \nu = -\frac{a}{d}y - \xi, \tag{3.167}$$

where

$$\xi = \tanh^{-1}\frac{x}{t}.$$

ν is periodic with period $8a$. We invert the relationships given in (3.167) to get (t, x, y) in terms of (τ, μ, ν) so that we can re-write the metric, and hence the wave equation (3.154), in terms of these new coordinates.

$$\xi = \frac{\mu - \nu}{2},\ y = -\frac{d}{a}(\mu + \nu),\ t = \tau\cosh(\frac{\mu-\nu}{2}), x = \tau\sinh(\frac{\mu-\nu}{2}). \tag{3.168}$$

Therefore:

$$
\begin{aligned}
dt &= d\tau\cosh\left(\frac{\mu-\nu}{2}\right) + \frac{1}{2}(d\mu - d\nu)\tau\sinh\left(\frac{\mu-\nu}{2}\right) &(3.169)\\
dx &= d\tau\sinh\left(\frac{\mu-\nu}{2}\right) + \frac{1}{2}(d\mu - d\nu)\tau\cosh\left(\frac{\mu-\nu}{2}\right) &(3.170)\\
dy &= -\frac{a}{2d}(d\mu + d\nu) &(3.171)
\end{aligned}
$$

In these coordinates the metric becomes

$$
\begin{aligned}
ds^2 &= d\tau^2 - \frac{\tau^2}{4}(d\mu - d\nu)^2 - \frac{d^2}{4a^2}(d\mu + d\nu)^2 \\
&= d\tau^2 - \frac{1}{4}(\tau^2 + \frac{d^2}{a^2})d\mu^2 - \frac{1}{4}(\tau^2 + \frac{d^2}{a^2})d\nu^2 + \frac{1}{2}(\tau^2 - \frac{d^2}{a^2})d\mu d\nu.
\end{aligned}
$$

In what follows put $\frac{a}{d} = A$. We get

$$g_{\tau\tau} = 1, \; g_{\mu\mu} = g_{\nu\nu} = -(\tau^2 + A^{-2}), \; g_{\mu\nu} = g_{\nu\mu} = \tau^2 - A^{-2}, \tag{3.172}$$

$$g^{\tau\tau} = 1, \; g^{\mu\mu} = g^{\nu\nu} = -(A^2 + \tau^{-2}), \; g^{\mu\nu} = g^{\nu\mu} = -(A^2 - \tau^{-2}). \tag{3.173}$$

All other components of the metric are zero and $det(-g) = \frac{\tau^2}{4A^2}$.

Now put the wave equation (3.154) in terms of these new coordinates, we get:

$$
\begin{aligned}
\frac{2A}{\tau}\Bigg[\partial_\tau\left(\frac{\tau}{2A}\partial_\tau\Phi\right) - \partial_\mu\left(\frac{\tau}{2A}\left(A^2 + \frac{1}{\tau^2}\right)\partial_\mu\Phi\right) - \partial_\nu\left(\frac{\tau}{2A}\left(A^2 + \frac{1}{\tau^2}\right)\partial_\nu\Phi\right) \\
-2\partial_\mu\partial_\nu\left(\frac{\tau}{2A}\left(A^2 - \frac{1}{\tau^2}\right)\Phi\right)\Bigg] = 0.
\end{aligned}
$$

This simplifies to give:

$$\partial_\tau^2\Phi + \frac{1}{\tau}\partial_\tau\Phi - (A^2 + \frac{1}{\tau^2})(\partial_\mu^2 + \partial_\nu^2)\Phi - 2(A^2 - \frac{1}{\tau^2})\partial_\mu\partial_\nu\Phi = 0. \tag{3.174}$$

Now Fourier analyse in our periodic coordinate ν by writing

$$\Phi(\tau, \mu, \nu) = \Re\left(\sum_k e^{iw_k\nu}\phi_k(\tau, \mu)\right). \tag{3.175}$$

Where $k \in \mathbf{Z}$ and

$$w_k = \frac{k\pi}{4a}. \tag{3.176}$$

Substituting back into (3.174) we get

$$\partial_\tau^2\phi_k + \frac{1}{\tau}\partial_\tau\phi_k - (A^2 + \frac{1}{\tau^2})\partial_\mu^2\phi_k - 2iw_k(A^2 - \frac{1}{\tau^2})\partial_\mu\phi_k + w_k^2(A^2 + \frac{1}{\tau^2})\phi_k = 0. \quad (3.177)$$

Consider a second order differential equation of the form:

$$au_{xx} + 2bu_{xy} + cu_{yy} + \ldots = 0. \quad (3.178)$$

If:

$b^2 - ac < 0$ the equation is elliptic,

$b^2 - ac > 0$ the equation is hyperbolic,

$b^2 - ac = 0$ the equation is parabolic.

In our case $b^2 - ac = (A^2 + \tau^{-2}) > 0$ and so our equation is hyperbolic. One can write the equation in characteristic form. The characteristics are the null geodesics of the symbol of our differential equation. They are solutions to the equation:

$$\frac{d\mu}{d\tau} = \pm\sqrt{A^2 + \frac{1}{\tau^2}}. \quad (3.179)$$

Integrating (3.179) gives

$$\mu + c = \pm\left[\sqrt{A^2\tau^2+1} + \frac{1}{2}\ln\left(\frac{\sqrt{A^2\tau^2+1}-1}{\sqrt{A^2\tau^2+1}+1}\right)\right], \quad (3.180)$$

as the equation of characteristics. They are the lines along which our scalar field propagates for this particular wave equation. Now define characteristic coordinates:

$$\lambda = \mu + \left[\sqrt{A^2\tau^2+1} + \frac{1}{2}\ln\left(\frac{\sqrt{A^2\tau^2+1}-1}{\sqrt{A^2\tau^2+1}+1}\right)\right], \quad (3.181)$$

$$\sigma = \mu - \left[\sqrt{A^2\tau^2+1} + \frac{1}{2}\ln\left(\frac{\sqrt{A^2\tau^2+1}-1}{\sqrt{A^2\tau^2+1}+1}\right)\right], \tag{3.182}$$

and writing the wave equation in terms of these coordinates gives:

$$\partial_\lambda\partial_\sigma\phi_k - \frac{A^2\tau^2}{4(A^2\tau^2+1)}(\partial_\lambda\phi_k - \partial_\sigma\phi_k) + \frac{iw_k}{2}\frac{A^2\tau^2-1}{A^2\tau^2+1}(\partial_\lambda\phi_k + \partial_\sigma\phi_k) - \frac{w_k^2}{4}\phi_k = 0. \tag{3.183}$$

(Note we cannot explicitly invert the above to get (τ, μ) in terms (λ, σ), but we regard τ and μ as known functions of λ and σ.

One can, in an analogous way, write down the wave equation in the non-chronal region. Here we put

$$t = \tau \sinh\xi, \;\; x = \tau\cosh\xi. \tag{3.184}$$

Again define coordinates

$$\mu = -\frac{a}{d}y + \xi, \;\; \nu' = -\frac{a}{d}y - \xi. \tag{3.185}$$

Under the Grant space isometries:

$$\mu \to \mu, \;\; \nu \to \nu - 8na. \tag{3.186}$$

We now re-write the wave equation in terms of these coordinates and we obtain:

$$\partial_\tau^2\Phi + \frac{1}{\tau}\partial_\tau\Phi + \frac{A^2}{\tau^2}(\tau^2 - A^{-2})(\partial_\mu^2\Phi + \partial_\nu^2\Phi) + \frac{2A^2}{\tau^2}(\tau^2 + A^{-2})\partial_\mu\partial_\nu\Phi = 0. \tag{3.187}$$

Once again put

$$\Phi(\tau, \mu, \nu) = \Re\left(\sum_k e^{iw_k\nu}\phi_k(\tau, \mu)\right). \tag{3.188}$$

Giving:

$$\partial_\tau^2\phi_k + \frac{1}{\tau}\partial_\tau\phi_k + \frac{A^2}{\tau^2}(\tau^2 - A^{-2})\partial_\mu^2\phi_k + \frac{2iw_kA^2}{\tau^2}(\tau^2 + A^2)\partial_\mu\phi_k - \frac{w_k^2A^2}{\tau^2}(\tau^2 - A^{-2})\phi_k = 0. \tag{3.189}$$

To determine the type of the equation we again calculate the determinant of the symbol.

$$b^2 - ac = -\frac{A^2}{\tau^2}(\tau^2 - A^{-2}). \tag{3.190}$$

So for:

$\tau^2 > A^{-2}$ the equation is elliptic,

$\tau^2 = A^{-2}$ the equation is parabolic,

$\tau^2 < A^{-2}$ the equation is hyperbolic.

So we have an equation of mixed type, with the equation changing signature over the surface $\tau^2 = x^2 - t^2 = A^{-2}$. What is the significance of this surface? We will return to this later.

In an identical method to the globally hyperbolic case we can calculate the characteristics for this differential equation in the hyperbolic region. These are given by:

$$\mu \pm c = \left[\sqrt{1 - A^2\tau^2} + \frac{1}{2}\ln\left(\frac{1 - \sqrt{1 - A^2\tau^2}}{1 + \sqrt{1 - A^2\tau^2}}\right)\right]. \tag{3.191}$$

Note that for $\tau^2 > A^{-2}$ we have no real characteristics. This corresponds to the fact that the equation become elliptic here. Define characteristic coordinates,

$$\lambda = \mu + \left[\sqrt{1 - A^2\tau^2} + \frac{1}{2}\ln\left(\frac{1-\sqrt{1-A^2\tau^2}}{1+\sqrt{1-A^2\tau^2}}\right)\right], \tag{3.192}$$

$$\sigma = \mu - \left[\sqrt{1 - A^2\tau^2} - \frac{1}{2}\ln\left(\frac{1-\sqrt{1-A^2\tau^2}}{1+\sqrt{1-A^2\tau^2}}\right)\right], \tag{3.193}$$

and re-write the wave equation in terms of these. We get:

$$\partial_\lambda\partial_\sigma\phi + \frac{A^2\tau^2}{4(1-A^2\tau^2)}(\partial_\lambda\phi - \partial_\sigma\phi) - \frac{iw_k}{2}\frac{1+A^2\tau^2}{1-A^2\tau^2}(\partial_\lambda\phi + \partial_\sigma\phi) - \frac{w_k^2}{4}\phi = 0. \tag{3.194}$$

So we have now constructed the wave equation in the chronal and non-chronal regions. However we cannot "glue" the equations together across the chronology horizon to get a wave equation for the whole spacetime. This is because in our coordinate system the wave equation becomes singular at $\tau = 0$. We now seek a coordinate system in which we can write down a wave equation for the whole spacetime. This is done as follows. In the chronal region define:

$$T = \tau\cosh\mu \tag{3.195}$$

$$X = \tau\sinh\mu \tag{3.196}$$

$$Y = y. \tag{3.197}$$

Where

$$\tau = \sqrt{t^2 - x^2},\ \mu = \xi - Ay. \tag{3.198}$$

So we can write T and X as follows:

$$T = \tau\cosh(-Ay + \xi)$$

$$
\begin{aligned}
&= \tau \cosh \xi \cosh Ay - \tau \sinh \xi \sinh Ay && (3.199)\\
&= t \cosh Ay - x \sinh Ay, &&\\
X &= \tau \sinh(-Ay + \xi) &&\\
&= \tau \sinh \xi \cosh Ay - \tau \cosh \xi \sinh Ay && (3.200)\\
&= x \cosh Ay - t \sinh Ay. &&
\end{aligned}
$$

Giving:

$$
\begin{aligned}
t &= T \cosh Ay + X \sinh Ay && (3.201)\\
x &= X \cosh Ay + T \sinh Ay && (3.202)\\
y &= Y. && (3.203)
\end{aligned}
$$

We can do a similar thing in the non-chronal region. Put

$$
\begin{aligned}
T &= \tau \sinh \mu && (3.204)\\
X &= \tau \cosh \mu && (3.205)\\
Y &= y, && (3.206)
\end{aligned}
$$

Where $\tau^2 = x^2 - t^2$ for the non-chronal region. We get:

$$
\begin{aligned}
T &= \tau \sinh(\xi - Ay) &&\\
&= \tau \sinh \xi \cosh Ay - \tau \cosh \xi \sinh Ay && (3.207)\\
&= t \cosh Ay - x \sinh Ay, &&\\
X &= \tau \cosh(\xi - Ay) &&\\
&= \tau \cosh \xi \cosh Ay - \tau \sinh \xi \sinh Ay && (3.208)
\end{aligned}
$$

$$= x\cosh Ay - t\sinh Ay.$$

Giving:

$$t = T\cosh Ay + X\sinh Ay \quad (3.209)$$

$$x = X\cosh Ay + T\sinh Ay \quad (3.210)$$

$$y = Y, \quad (3.211)$$

as in the chronal region. So we now have a new coordinate system (T, X, Y) with which we can cover the entire spacetime. The coordinates are perfectly well behaved at the chronology horizon, which is located at $T^2 - X^2 = 0$. In this coordinate system T and X are invariant under the Grant space isometries and Y becomes the periodic coordinate, with period $4nd$. We can write the wave equation in terms of T, X, Y now and Fourier analyse in the Y coordinate.

First re-write the metric in terms of T, X, Y:

$$t = T\cosh AY + X\sinh AY \quad (3.212)$$

$$dt = (dT + AXdY)\cosh AY + (dX + ATdY)\sinh AY \quad (3.213)$$

$$x = X\cosh AY + T\sinh AY \quad (3.214)$$

$$dx = (dT + AXdY)\sinh AY + (dX + ATdY)\cosh AY. \quad (3.215)$$

Giving metric $\mathbf{g}'$

$$ds^2 = (dT + AXdY)^2 - (dX + ATdY)^2 - dY^2. \quad (3.216)$$

Expanding this we get:

$$ds^2 = dT^2 - dX^2 - (1 + A^2(T^2 - X^2))dY^2 + 2AXdTdY - 2ATdXdY. \quad (3.217)$$

From the above we can see that the points of the orbit become null separated when

$$X^2 - T^2 = A^{-2},$$

and they become timelike for

$$X^2 - T^2 > A^{-2}.$$

The non-zero components of g'^{ab} are given by:

$$g^{TT} = 1 - A^2X^2,\ g^{xx} = -(1 + A^2T^2),\ g^{YY} = -1, \quad (3.218)$$
$$g^{TX} = g^{XT} = -A^2TX,\ g^{TY} = g^{YT} = AX,\ g^{XY} = g^{YX} = AT,$$

and $det(-g) = 1$.

We can now write down the wave equation in the (T, X, Y) coordinates.

$$\begin{aligned}\partial_T([1 - A^2X^2])\partial_T\Phi) &- \partial_T(A^2TX\partial_X\Phi) - \partial_X(A^2TX\partial_T\Phi) - \partial_X([1 + A^2T^2]\partial_X\Phi) \\ &+ 2AX\partial_T\partial_Y\Phi + 2AT\partial_X\partial_Y\Phi - \partial_Y^2\Phi = 0. \quad (3.219)\end{aligned}$$

As any solution to the wave equation in Grant space will now be periodic in Y we consider solutions of the form:

$$\Phi(T, X, Y) = \Re(\sum_k e^{iw_k y}\phi_k(T, X)), \quad (3.220)$$

where $w = \frac{\pi k}{2d}$. The wave equation (3.219) becomes:

$$\begin{aligned}(1 - A^2X^2)\partial_T^2\phi_k \;&-\; (1 + A^2T^2)\partial_X^2 - 2A^2TX\partial_T\partial_X\phi_k - A^2X\partial_X\phi_k - A^2T\partial_T\phi_k \\ &+\; 2Aiw[T\partial_X\phi_w + X\partial_T\phi_w] + w^2\phi_w = 0. \end{aligned} \tag{3.221}$$

We will call (3.221) the reduced wave equation. Now we can look at the characteristics of this equation. Again consider the symbol of the differential equation, h^{ab}:

$$h^{ab} = \begin{pmatrix} 1 - A^2X^2 & -A^2TX \\ -A^2TX & -1 - A^2T^2 \end{pmatrix}. \tag{3.222}$$

$det(h) = -[1 + A^2(T^2 - X^2)]$ so inverting the above gives:

$$h_{ab} = -\frac{1}{1 + A^2(T^2 - X^2)} \begin{pmatrix} -1 - A^2T^2 & A^2XT \\ A^2XT & 1 - A^2X^2 \end{pmatrix}. \tag{3.223}$$

The characteristics are null geodesics of h_{ab}, and are therefore given by solutions of:

$$\frac{dT}{dX} = \frac{A^2XT \pm \sqrt{1 + A^2(T^2 - X^2)}}{1 + A^2T^2}. \tag{3.224}$$

Clearly we cannot solve the differential equation (3.224) as it stands, however we can write the characteristics obtained in the (τ, μ) coordinates in terms of the (T, X) coordinates and verify that they do in fact satisfy the equation (3.224). In the chronal region

$$\tau^2 = T^2 - X^2, \;\; \mu = \frac{1}{2}\ln\left(\frac{T + X}{T - X}\right), \tag{3.225}$$

and in the non-chronal region

$$\tau^2 = X^2 - T^2, \;\; \mu = \frac{1}{2}\ln\left(\frac{T + X}{X - T}\right). \tag{3.226}$$

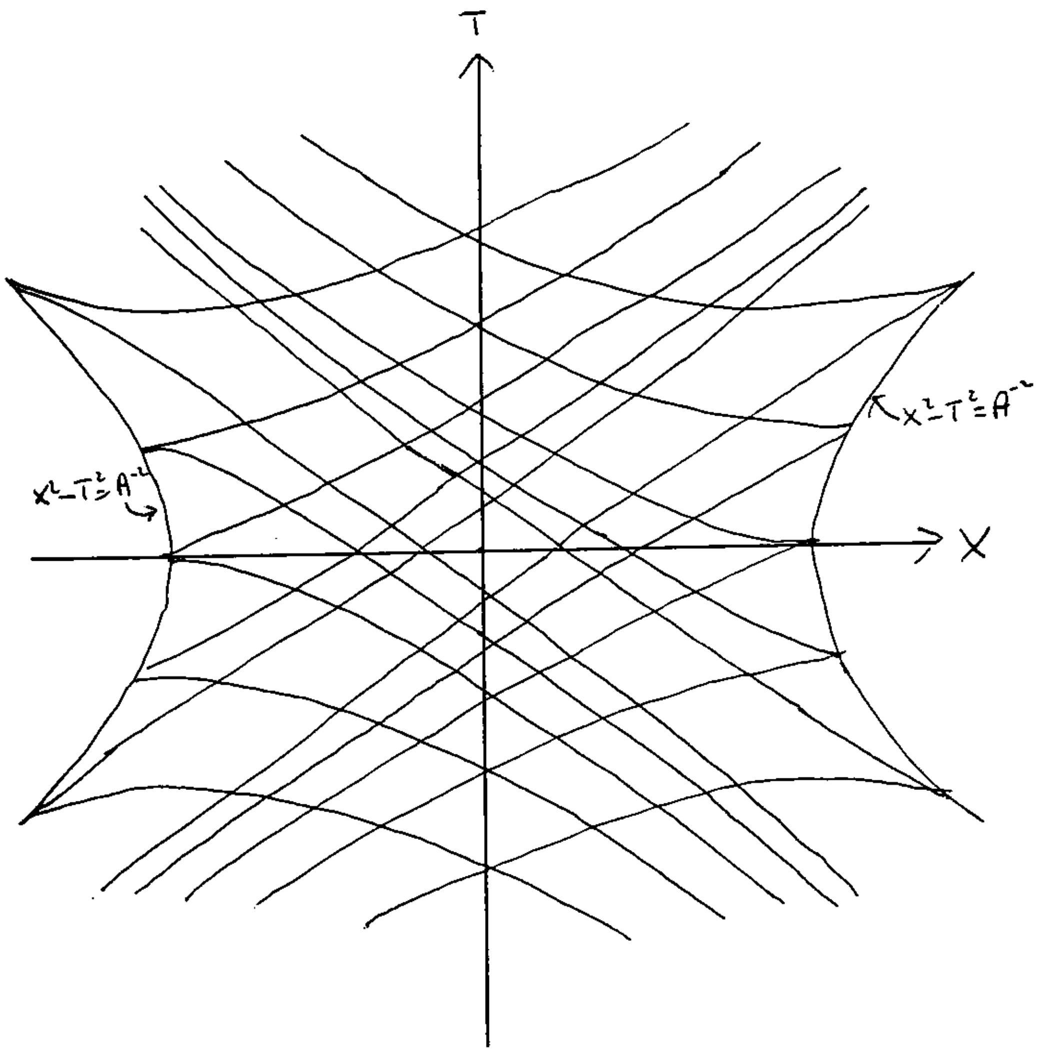

Figure 3.8: *The characteristics of the reduced wave equation.*

So in our T, X coordinates the characteristics are:

$$\frac{1}{2}\ln\left[\left(\frac{T+X}{T-X}\right)\left(\frac{\sqrt{A^2(T^2-X^2)+1}\mp 1}{\sqrt{A^2(T^2-X^2)+1}\pm 1}\right)\right] \pm \sqrt{A^2(T^2-X^2)+1} = c, \qquad (3.227)$$

where c is a constant. A plot showing the characteristics is given in figure (3.8).

We see that there exist real characteristics only for $X^2 - T^2 \leq A^{-2}$, as one would expect because outside that region the reduced wave equation becomes elliptic.

We can conformally flatten h_{ab} in the hyperbolic region by defining new coordinates

$$u = \frac{1}{2}(\lambda + \sigma), \ v = \frac{1}{2}(\lambda - \sigma), \tag{3.228}$$

where λ and σ are the characteristic coordinates. This gives

$$\begin{aligned} u &= \frac{1}{2}\ln\left(\frac{T+X}{T-X}\right), && (3.229) \\ v &= \frac{1}{2}\ln\left(\frac{\sqrt{A^2(T^2-X^2)+1}-1}{\sqrt{A^2(T^2-X^2)+1}+1}\right) + \sqrt{A^2(T^2-X^2)+1}, && (3.230) \end{aligned}$$

and

$$\begin{aligned} du &= \frac{TdX - XdT}{T^2 - X^2}, && (3.231) \\ dv &= \frac{TdT - Xdx}{T^2 - X^2}\sqrt{A^2(T^2-X^2)+1}. && (3.232) \end{aligned}$$

So h_{ab} becomes

$$ds^2 = f(T,X)(du^2 - dv^2), \tag{3.233}$$

where the conformal factor is

$$f(T,X) = \frac{X^2 - T^2}{1 + A^2(T^2 - X^2)}. \tag{3.234}$$

A look at the plot of characteristics shows that in the two dimensional problem there are certain spacelike surfaces that the characteristics cross only once — even in the non-chronal region. These surfaces have a well defined domain of dependence, and so can be used as partial Cauchy surfaces (in the two-dimensional problem) upon which we can give initial data. In the chronal region the surfaces $T^2 - X^2 = C^2$, where C is a constant, can be used as Cauchy surfaces. These are just the $t^2 - x^2 = C^2$ surfaces in Minkowski space.

For the time being we will focus our attention on the surface $T = 0$, $-A^{-1} < X < A^{-1}$. We will then see what this surface looks like in Minkowski space, the aim being to see if we can relate what is happening in the two-dimensional problem to what is happening in the full three-dimensional problem. The surface $T = 0$ becomes

$$t \cosh Ay - x \sinh Ay = 0 \Rightarrow t = x \tanh Ay. \tag{3.235}$$

when written in Minkowski coordinates, and the condition $X^2 < A^{-2}$ becomes

$$X^2 < A^{-2} \Rightarrow T^2 + X^2 - T^2 < A^{-2}, \tag{3.236}$$

but $T = 0$ and $X^2 - T^2 = x^2 - t^2$, so we get

$$x^2 - t^2 < A^{-2}. \tag{3.237}$$

So the $T = 0$ surface in Minkowski space takes the form:

$$y = \frac{1}{2A} \ln\left(\frac{x+t}{x-t}\right), \tag{3.238}$$

when $x^2 - t^2 < A^{-2}$. Note that our surface can also be re-written:

$$x^2 - t^2 = x^2 \operatorname{sech}^2 Ay, \tag{3.239}$$

so our surface only lies in the region $x^2 - t^2 \geq 0$, the non-chronal region. The surface is spacelike when

$$1 - \tanh^2 Ay - A^2 x^2 \operatorname{sech}^4 Ay \quad < \quad 0$$

$$\Rightarrow \operatorname{sech}^2 Ay(1 - A^2x^2 \operatorname{sech}^2 Ay) \;<\; 0$$
$$\Rightarrow x^2 - t^2 \;<\; A^{-2}.$$

So $T = 0,\ x^2 - t^2 < A^{-2}$ is spacelike everywhere. Note however for values of t and x,where $x^2 - t^2 > A^{-2}$, the surface $T = 0$ becomes timelike. In the two-dimensional picture we have seen that free data can be prescribed on $T = 0$ for $A^2X^2 < 1$, and the evolution of this data has a well defined domain of dependence. Since the corresponding surface in the three-dimensional problem is spacelike one might expect that it too would be a partial Cauchy surface. However this is not the case. Recall that for $x^2 - t^2 < A^{-2}$ our spacetime contains self intersecting null geodesics. If a point (t, x, y) lies in the surface $t = x \tanh Ay$, then so does its n^{th} image point $(t \cosh 4na + x \sinh 4na, x \cosh 4na + t \sinh 4na, y + 4nd)$ (as our surface is invariant under the Grant space isometries). For

$$x^2 - t^2 = \frac{8n^2d^2}{\cosh 4na - 1}, \tag{3.240}$$

we can join the point (t, x, y) to it n^{th} image point by a null curve, and so therefore there exists null curves that cross $T = 0$ more than once. Thus even though the surface is spacelike it is not achronal and cannot be a partial Cauchy surface.

We now return to our investigation of the surface $x^2 - t^2 = A^{-2}$. Recall the expression for a point (t, x, y) to lie on the n^{th} polarised hypersurface is given by:

$$x^2 - t^2 = \frac{8n^2d^2}{\cosh 4na - 1} = \frac{4n^2d^2}{\sinh^2 2na}. \tag{3.241}$$

Taking the $n \to 0$ limit of this gives

$$x^2 - t^2 = \frac{d^2}{a^2} = A^{-2}. \tag{3.242}$$

For this reason we call the surface the "0^{th}-polarised hypersurface". Note also that on this surface the points on the orbits of the Grant space isometries become null separated. We will look at the significance of this surface further in the next chapter when we study the spinning cosmic string spacetime, and note that there appears to be similarities between this spacetime and the non-chronal region of Grant space.

Sobolev Spaces and the Fourier Approach

In this section we will look at combining the Fourier approach with the Sobolev spaces we constructed in the previous section. We will verify that given initial data in the chronal region on a surface $t^2 - x^2 = \tau_0^2$ the exists a unique solution to the wave equation up to and including the chronology horizon. The weighted Sobolev spaces constructed in the previous section will be modified, and written in the (T, X, Y) co-ordinates. Using the fact that data is periodic in Y we can construct two-dimensional function spaces for our initial data.

A function $f(t, x, y)$ (where f can be either initial data, or the solution to the wave equation) invariant under the Grant space isometries can be written as the sum of functions of the form

$$\Re(e^{iwY} F_w(T, X)).$$

We will now relate the function spaces in the (t, x, y) and (T, X, Y) coordinates.

We are looking at the Cauchy problem in the chronal region with data given on the surface $t^2 - x^2 = \tau_0^2$. We will need to convert the function spaces from (t, x, y) coordinates to (T, X, Y) coordinates. First we convert the normal and tangential derivatives to S_{τ_0} into the (T, X, Y) coordinates. Recall

$$\begin{aligned} T &= t \cosh Ay - x \cosh Ay \\ X &= x \cosh Ay - t \sinh Ay \\ Y &= y. \end{aligned}$$

So we have

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial T}{\partial t}\frac{\partial}{\partial T} + \frac{\partial X}{\partial t}\frac{\partial}{\partial X} \\ &= \cosh Ay \frac{\partial}{\partial T} - \sinh Ay \frac{\partial}{\partial X}, \\ \frac{\partial}{\partial x} &= \frac{\partial T}{\partial x}\frac{\partial}{\partial T} + \frac{\partial X}{\partial x}\frac{\partial}{\partial X} \\ &= \cosh Ay \frac{\partial}{\partial X} - \sinh Ay \frac{\partial}{\partial T}. \end{aligned}$$

Now the normal and tangential derivatives to the surface $t^2 - x^2 = \tau_0^2$ are given by

$$\begin{aligned} \frac{\partial}{\partial n} &= -\frac{t}{\sqrt{t^2 - x^2}}\frac{\partial}{\partial t} - \frac{x}{\sqrt{t^2 - x^2}}\frac{\partial}{\partial x}, \\ \frac{\partial}{\partial \tau} &= -\frac{x}{\sqrt{t^2 - x^2}}\frac{\partial}{\partial t} - \frac{t}{\sqrt{t^2 - x^2}}\frac{\partial}{\partial x}, \end{aligned}$$

converting to $(T,\ X,\ Y)$ coordinates they become

$$\begin{aligned} \frac{\partial}{\partial n} &= -\frac{T}{\sqrt{T^2 - X^2}}\frac{\partial}{\partial T} - \frac{X}{\sqrt{T^2 - X^2}}\frac{\partial}{\partial X}, \\ \frac{\partial}{\partial \tau} &= -\frac{X}{\sqrt{T^2 - X^2}}\frac{\partial}{\partial T} - \frac{T}{\sqrt{T^2 - X^2}}\frac{\partial}{\partial X}, \end{aligned}$$

We see that the normal and tangential derivatives take the same form in both sets of coordinates.

Now consider the volume element in the (T, X, Y) coordinates. On the surface S_{τ_0} we have

$$dS = \frac{\tau_0}{x^2+\tau^2}dxdy = \frac{\tau_0}{X^2+\tau^2}dXdY.$$

As the normal and tangential derivatives, and the volume element on S_{τ_0} take the same form in both the (t, x, y) coordinates and the (T, X, Y) coordinates then our weighted energy Sobolev norm is invariant under the change of coordinates as well.

Let Φ_0 and Φ_1 be initial data for the wave equation on S_{τ_0}. For Φ_0 and Φ_1 to represent initial data on Grant space they can be written as a finite sum of modes as follows:

$$\Phi_0(t, x, y) = \Re(\sum_k e^{iw_k Y}\phi_{0_k}(T, X)), \tag{3.243}$$

$$\Phi_1(t, x, y) = \Re(\sum_k e^{iw_k Y}\phi_{1_k}(T, X)). \tag{3.244}$$

Definition 3.18 *Let $W^{E_F}_{m,\ \alpha}(S_{\tau_0})$ be the reduced weighted energy Sobolev space with norm $\|\phi_k, S_{\tau_0}\|^{E_F}_m$ given by*

$$\|\phi_k, S_{\tau_0}\|^{E_F}_m = \int_{S_{\tau_0}} \left(|\sum_k \phi_k|^2 + \sum_{i=1}^{m}|\sum_k w_k^m D^i\phi_k|^2\right)\alpha^{2m-1}dS, \tag{3.245}$$

where dS is given by

$$dS = \frac{\tau_0}{X^2+\tau_0^2}dX,$$

and derivatives taken in directions tangential to S_{τ_0} in the $Y =$ constant plane.

Theorem 3.19 *Let ϕ_{0_w} and ϕ_{1_w} be such that*

$$\phi_{0_w} \in W^{E_F}_{m+1,\ \alpha}(S_{\tau 0}),\ \ \phi_{1_w} \in W^{E_F}_{m,\ \alpha}(S_{\tau 0}),$$

for every w where $\phi_{i_w} \neq 0$ then there exists a unique solution $\Phi \in W^m(V)$ to the wave equation.

Proof

In the following let i be 0 or 1. We will prove our result as a series of lemmas. In what follow the k_i will be positive constants.

Lemma 3.20

$$\|\Phi_i, S_{\tau 0}\|^E_m \leq k_1 \|\phi_{w_i}, S_{\tau 0}\|^{E_F}_m. \tag{3.246}$$

Proof

$$\|\Phi_i, S_{\tau 0}\|^E_m = \|\Re(\sum_w e^{iwY} \phi_{w_i}(T, X)), S_{\tau 0}\|^E_m.$$

Now as the m^{th} Y derivative of Φ_i is simply a multiple of $w^m \Phi_i$, we get

$$\|\Phi_i, S_{\tau 0}\|^E_m \leq k_1 \int_{-d}^{d} |e^{iwY}| dY \|\phi_{w_i}(T, X), S_{\tau 0}\|^{E_F}_m.$$

As the Y range of integration is over a finite domain we can bound the above by

$$\|\Phi_i, S_{\tau 0}\|^E_m \leq k_2 \|\phi_{w_i}(T, X), S_{\tau 0}\|^{E_F}_m.$$

So we see that provided

$$\phi_{w_i} \in \|\phi_{w_i}(T, X), S_{\tau_0}\|_m^{E_F},$$

then

$$\Phi_i \in \|\Phi_{w_i}(T, X)), S_{\tau_0}\|_m^{E}.$$

Therefore if

$$\phi_{0_w} \in W^{E_F}_{m+1,\ \alpha}(S_{\tau_0}),\ \ \phi_{1_w} \in W^{E_F}_{m,\ \alpha}(S_{\tau_0})$$

then

$$\Phi_0 \in W^{E}_{m+1,\ \alpha}(S_{\tau_0}),\ \ \Phi_1 \in W^{E}_{m,\ \alpha}(S_{\tau_0})$$

so using theorem (3.14) we have proven our theorem.

□

Data in the non-chronal region

We will now prove existence and uniqueness of solutions to the wave equation in the non-chronal region of Grant space given initial data on the surface $T = 0$, $0 < X < A^{-1}$ in the hyperbolic region. We will do this by proving existence of solutions to the Fourier reduced wave equation (3.221). In this two dimensional problem we are only interested in local existence and may therefore assume the initial data is of compact support so we can use results from Hawking and Ellis pages $237 - 243$ without having to employ the use of weighted Sobolev spaces as we did in the previous section when we considered initial data in the chronal region. Because of the subtle difference between the work in [HE] and that done at the beginning of this chapter we will introduce the definitions of the Sobolev spaces we need here.

Let M be some two dimensional manifold and let $\widehat{\mathbf{g}}$ be a background metric on M. For simplicity with take this to be the two dimensional Minkowski space metric. We also introduce a positive definite metric e_{ab} on M. Let N be an embedded manifold in M. Then $\|\phi, N\|_m$ is defined to be

$$\left(\sum_{p=0}^{m} \int_N (D^p\phi)^2 d\sigma\right)^2, \tag{3.247}$$

where $d\sigma$ is the volume element on N induced by $\mathbf{e}$. $D^p\phi = \phi;_L$ where $|L| = p$. Here ; denotes the covariant with respect to $\widehat{\mathbf{g}}$.

The *Sobolev spaces* $W^m(N)$ are defined to be the vector spaces of scalar fields ϕ whose values and derivatives are defined almost everywhere on N and for which $\|\phi, N\|_m$ is finite.

Let s_0 denote the initial data surface $T = 0$, $0 \leq X \leq X_0$. Let ϕ_0 and ϕ_1 be initial data in the form of the values of ϕ and its normal derivative on s_0 respectively. s_0 and D^+s_0 both have compact closure in M. In what follows all covariant derivatives are with respect to $\widehat{\mathbf{g}}$.

Theorem 3.21 *Let*

$$\begin{aligned} \phi_0 &\in W^{4+a}(s_0), \\ \phi_1 &\in W^{3+a}(s_0), \end{aligned}$$

then there exists a unique solution $\phi \in W^{4+a}(D^+(s_0))$ *of (3.221), the reduced wave equation in Grant space.*

Proof

The reduced wave equation (3.221) takes the form

$$h^{ab}\phi_{;ab} + B^a\phi_{;a} + C\phi = 0.$$

Existence of solutions to the reduced wave equation follows from Proposition 7.4.7 of [HE], provided **h**, **B** and **C** satisfy certain conditions stated below.

To make use of the results in [HE] we require that on $\overline{D^+(s_0)}$ there exists some $Q_1 > 0$ such that

$$h_{ab}n^a n^b \geq Q_1, \tag{3.248}$$

and

$$h_{ab}W^a W^b \leq -Q_1 e_{ab} W^a W^b, \tag{3.249}$$

for any form **W** which satisfies $h_{ab}n^a W^b = 0$. In the above **e** is a positive definite metric on M and **n** is the normal vector to s_0.

We also require that there exists a constant Q_2 such that

$$\|\mathbf{A}, D^+(s_0)\|_{4+a} < Q_2, \ \|\mathbf{B}, D^+(s_0)\|_{3+a} < Q_2, \ \|\mathbf{C}, D^+(s_0)\|_{3+a} < Q_2. \tag{3.250}$$

It is also necessary that $\overline{D^+(s_0)}$ is achronal with respect to **h**, which indeed it is.

Consider condition (3.248) for the surface s_0

$$h_{ab}n^a n^b = (1 - A^2X^2) \geq (1 - A^2X_0^2) = Q_1.$$

Now consider condition (3.249), and consider forms $\mathbf{W}$ that satisfying

$$h_{ab}n^a W^b = 0. \tag{3.251}$$

We get

$$h_{ab}n^a W^b = W^T,$$

so for $\mathbf{W}$ to satisfy (3.251) we require

$$W^T = 0,\ W^X = C,$$

where C is a constant. Now

$$\begin{aligned} h_{ab}W^a W^b &= -\frac{1 - A^2X^2)}{1 + A^2(T^2 - X^2)}C^2 \\ &\leq -(1 - A^2X^2)C^2 \\ &\leq -(1 - A^2X_0^2)C^2 \\ &= -Q_1C^2 \\ &= -Q_1 e_{ab}W^a W^b, \end{aligned}$$

where $e_{ab} = \delta_{ab}$. So condition (3.249) is satisfied.

Notice in the above that Q_1 approaches zero as X_0 approaches A^{-1}. This is where the Cauchy surface $T = 0$ fails to remain spacelike with respect to $\mathbf{h}$.

Condition (3.250) is satisfied as the values of **h**, **B**, **C** and their derivatives with respect to T and X all remain finite on $D^+(s_0)$, and $D^+(s_0)$ has compact closure in M.

As all the above conditions are satisfied we can deduce the existence of a solution ϕ to the reduced wave equation given initial data on s_0. from Proposition 7.4.7 of [HE].

□

We have shown that there exits solutions to the reduce wave equation given initial data on s_0. We can, therefore construct a general solution to the full wave equation on Grant space as a finite sum of the modal solutions to the reduced equation.

$$\Phi(T, X, Y) = \Re(\sum_w e^{iwy}\phi_w(T, X)).$$

We have seen that Gott space and Grant space have well defined Cauchy problems for the scalar wave equation given initial data on surfaces in both the chronal region and part of the non-chronal region. However there is a surface over which the reduced wave equation changes type, and this shows that one does not have a well defined Cauchy problem in this region. We will consider this further in the next chapter when we consider the related problem in the spinning cosmic string spacetime.

3.3 The Cauchy Problem on Wormholes spacetimes

In this section we look at work done by Friedman et al. [FMN] on the Cauchy problem in the twin paradox wormhole spacetimes.

We again examine the behaviour of a test scalar field propagating on a background spacetime. The wormhole spacetime is flat everywhere except near the wormhole's mouths.

In what follows we will consider the geometric optics approximation for wave propagation, taking the frequency of the waves (w) to be large compared to $\frac{1}{b}$, where b is the radius of the wormhole mouth. In the geometric optics limit the scalar field will propagate along null geodesics. These null geodesics will be the characteristics of the wave equation in the wormhole spacetime, so to get an intuitive idea of what will happen when we look at the Cauchy problem we first consider tracing null geodesics through the wormhole.

Consider a geodesic incident on the wormhole mouth 2 in the $\phi = 0$ plane, with the frequency of the scalar field such that $wb \gg 1$. Due to the identifications on the wormhole mouths, the wave fronts will emerge at the positions from mouth 1 shown in figure (3.10). In the geometric optics limit waves propagate in a direction orthogonal to the wave front. We can see that the wave spreads as it emerges from the wormhole, see figure (3.11). Consider now the focal length of the wormhole. This will be the distance behind the wormhole mouth where the rays would appear to converge if continued backwards. We can use some elementary geometry to calculate

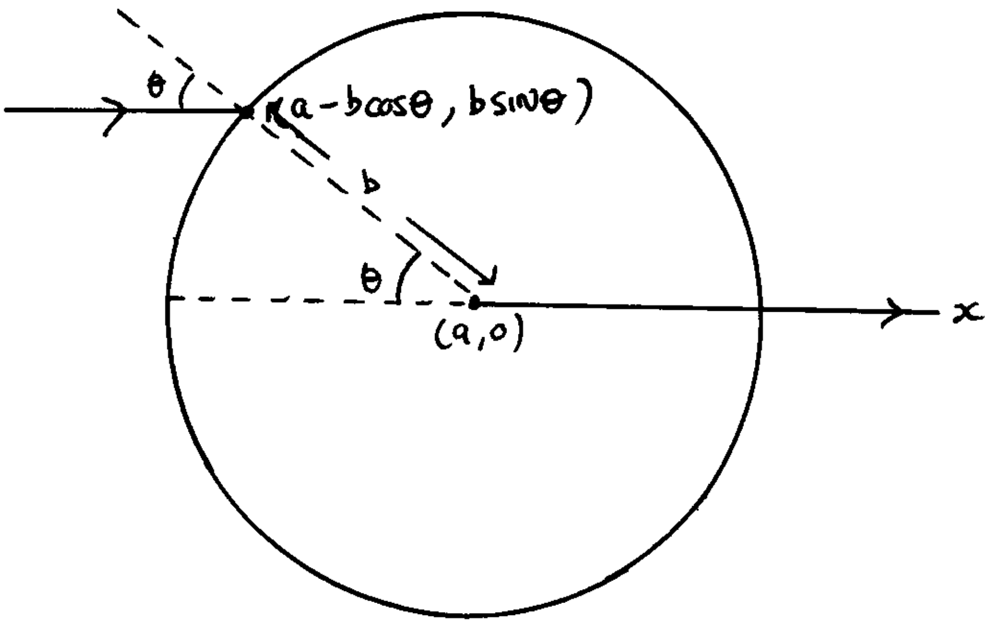

Figure 3.9: *The ray entering the righthand wormhole mouth.*

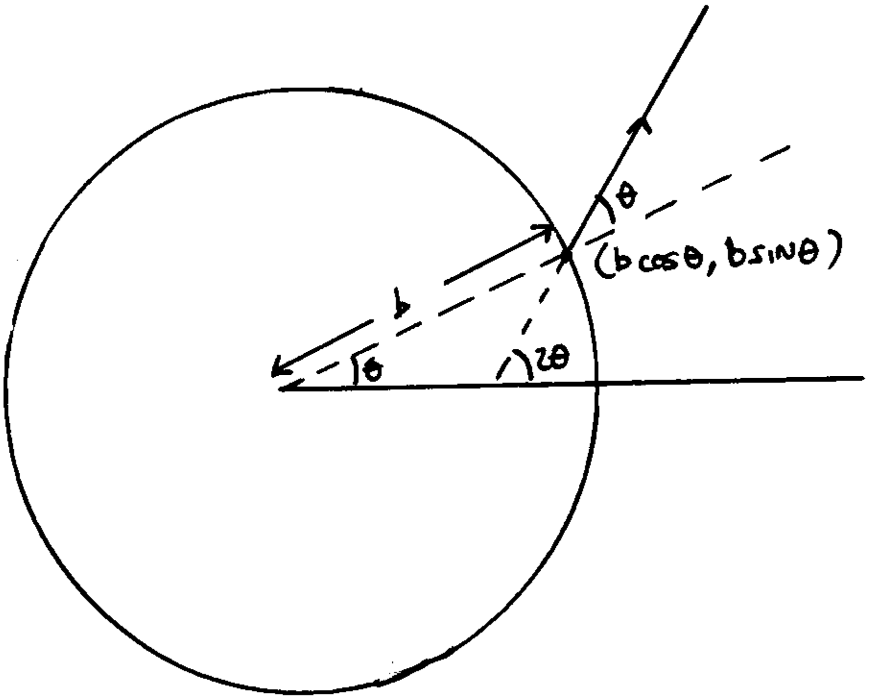

Figure 3.10: *The ray emerging from the lefthand wormhole mouth.*

the focal length. Consider a ray parallel to the x−axis which enters the righthand wormhole mouth at a point $(a - b\cos\theta, b\sin\theta)$. Then as a result of the identification made, the equation of the emerging ray is

$$x \sin 2\theta - y \cos 2\theta = b \sin \theta, \tag{3.252}$$

see figures (3.9) and (3.10). Two neighbouring rays intersect at the point given by solving the simultaneous equations

$$\begin{aligned} x \sin 2\theta - y \cos 2\theta &= b \sin \theta, \\ 2x \cos 2\theta + 2y \sin 2\theta &= b \cos \theta, \end{aligned}$$

i.e. at the point

$$x = \frac{1}{2}b + b \sin^2 \theta \cos \theta, \ \ y = b \sin^3 \theta.$$

Thus for rays close to the $x-$axis the wormhole acts like a diverging lens with focal length $\frac{b}{2}$.

We are now ready to consider the stability of the chronology horizon in the wormhole spacetime. The most likely place for an instability is on the closed null geodesic. This is because all null geodesics pile up on the closed null geodesic. Also any field propagating along the closed null geodesic travels through the wormhole an infinite number of times, so any blue shifting effects of the wormhole will be most apparent here. Consider a field propagating round the wormhole. Recall that the wormhole acts as a diverging lens, see figure (3.11) . This is due to the violation of the weak energy condition needed to form the wormholes — the weak energy condition plus Einstein's field equations imply the convergence of null geodesic, so the divergence of the geodesics requires the violation of the weak energy condition. As the field

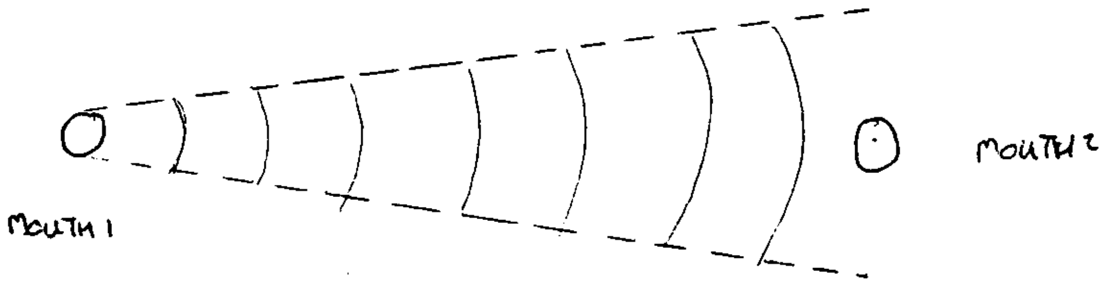

Figure 3.11: *A null ray being spread by the wormhole.*

leaves mouth 1 it is spread laterally (see figure (3.11) , and so only a fraction of the wave enters mouth 2. So with each traversal of the wormhole the amplitude of the wave entering mouth 2 is reduced. To see how much the amplitude is reduced consider the proportion of radiation which enters the righthand mouth. Since $b \ll a$, any ray which enters the righthand mouth subtends an angle less than $\phi \approx \frac{b}{a}$ (see figure (3.12)). However this means that it entered the mouth at most a distance $h = b \sin \frac{\phi}{2}$ from the axis (see figure (3.13)). Thus using $b \ll a$ we have $h \approx \frac{b^2}{2a}$. Hence a beam of radiation of cross-sectional area

$$\pi \left(\frac{b^4}{4a^2} \right)$$

eventually enters the righthand mouth of cross-sectional area πb^2. The ratio of these areas is

$$\left(\frac{b}{2a} \right)^2 .$$

Hence the amplitude is diminished by a factor of $\frac{b}{2a}$ on each traversal of the wormhole.

However for a moving wormhole the energy of the beam is also affected by the change in frequency between entering the righthand mouth and leaving the lefthand

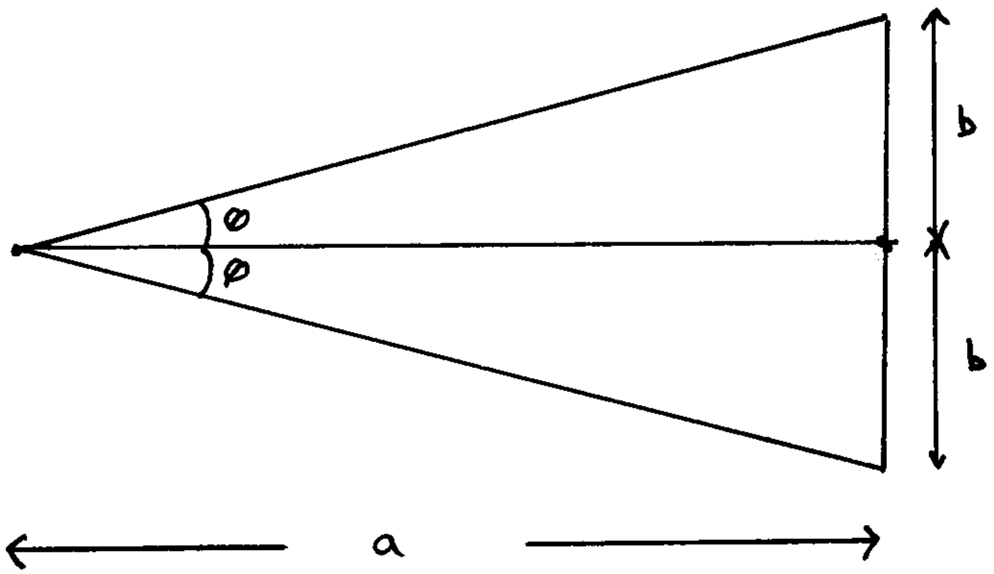

Figure 3.12: *The proportion of the wave that leaves mouth 1 and enters mouth 2.*

mouth. For an observer in Minkowski space moving towards a stationary source with speed v (in natural units) the frequency is blue shifted by a factor B where

$$B = \sqrt{\frac{1+v}{1-v}}. \tag{3.253}$$

Hence the frequency of the radiation emitted from the lefthand mouth is blueshifted by this amount where v is the speed at which the righthand mouth is moving towards the lefthand mouth when the CTC is formed. However provided

$$C = \frac{b}{2a}\sqrt{\frac{1+v}{1-v}} < 1, \tag{3.254}$$

the decrease in energy due to the change in amplitude will dominate the increase in

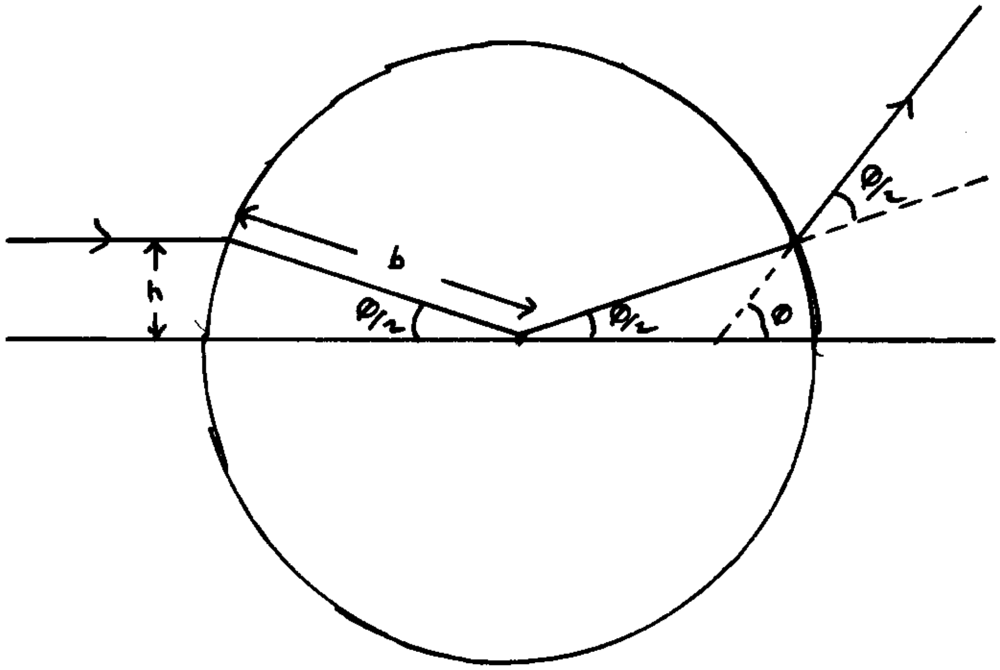

Figure 3.13: *The ray entering mouth 1 at a distance h above the x−axis.*

energy due to the blueshift. Furthermore

$$\sum_{n=0}^{\infty} C^n < \infty,$$

so the total energy remains finite. Thus it appears that the diverging lens property of the wormhole stabilises the chronology horizon - the reduction in amplitude of the wave counteracts the blue shifting. Thus it would appear that the chronology horizon in the wormhole spacetime is immune to the type of instabilities that affect Misner space.

Now we have shown that the horizon $\mathcal{H}$ appears to be stable, what can be said about the evolution of the field through and beyond $\mathcal{H}$? Let us examine the Cauchy problem with data posed at past null infinity ($\mathcal{I}^-$). Now one might think that data at $\mathcal{I}^-$ would not be sufficient to determine the full evolution of Φ. This is because the characteristics for the wave equation are future directed null geodesics, and not all such geodesics originate on $\mathcal{I}^-$. For example we may have data entering the spacetime from from the closed null geodesic on $\mathcal{H}$. However data cannot enter the spacetime from this route because geodesics that originate from the CNG spiral round the wormhole an infinite number of times as they peel off the CNG, and in doing so their amplitude is driven to zero. Thus the same mechanism that stabilises $\mathcal{H}$ prevents new data from entering the spacetime from the CNG. There are also future directed null geodesics that enter the spacetime from future timelike infinity, and propagate down to $\mathcal{H}$ via an infinite number of wormhole traversals, but again the amplitudes of fields travelling along these geodesics is driven to zero, preventing new data from entering the spacetime from this route.

In a spacetime with no CTCs the full spacetime development can be uniquely determined given initial data on $\mathcal{I}^-$. Do the CTCs present, and the "principle of self consistency" constrain the data in any way? Thorne et al. [FMN] suggest that the data is not constrained in any way. Consider the scalar field Φ propagating in this spacetime. It propagates as in flat space until it encounters the wormhole mouth. At this point some of the wave is transmitted and some reflected. The transmitted field then continues to propagates, linearly superposing on the field that has followed other routes.

To keep track of what is going on we define the following:

$\Phi^{(0)}$ is the incident field that does not under go scattering.

$\Phi_1^{(1)}$ is the field that has scattered once and is outgoing from mouth 1 (it consists of a transmission of $\Phi^{(0)}$ into mouth 2 , and a refection of $\Phi^{(0)}$ out of mouth 1).

$\Phi_2^{(1)}$ is as $\Phi_1^{(1)}$ with the roles of mouth 1 and 2 reversed.

Each component $\Phi_i^{(k)}$ originates at mouth i and is fed in by a transmission of $\Phi_i^{(k-1)}$ and a reflection of $\Phi_j^{(k-1)}$ $(i, j = 1, 2, i \neq j)$. This field then propagates — some going down mouth j and the rest going to future null infinity. At any event in the spacetime the full field Φ is given by:

$$\Phi = \sum_{k=1}^{\infty} (\Phi_1^{(k)} + \Phi_2^{(k)}). \tag{3.255}$$

If this sum converges for all initial data then it appears our initial data need not be constrained. On each trip round the wormhole the amplitude of high frequency fields is reduced by a factor $\frac{b}{2a}$. It seems reasonable (and apparently has been checked by computer for k up to 10^6 [FMN]) that Φ is a convergent power series in $\frac{b}{2a}$.

Thus it appears that for every choice of initial data in $\mathcal{I}^-$ there exists a unique solution to the wave equation. However the solution Φ may not be smooth at $\mathcal{H}$. Suppose that Φ is of the form

$$\Phi = Af(w(t, \mathbf{r})). \tag{3.256}$$

After m traversals of the wormhole:

$$\Phi \rightarrow \left(\frac{b}{2a}\right)^m Af\left(\left(\sqrt{\frac{1+v}{1-v}}\right)^m w(t, \mathbf{r})\right). \tag{3.257}$$

The n^{th} derivative of Φ will be

$$\left(\frac{b}{2a}\right)^m \left(\sqrt{\frac{1+v}{1-v}}\right)^{mn} f^{(n)}\left(\left(\sqrt{\frac{1+v}{1-v}}\right)^m (t,\mathbf{r})\right). \tag{3.258}$$

Now to ensure this is finite for all m (as $m \to \infty$ on approaching $\mathcal{H}$). We require

$$\frac{b}{2a}\left(\sqrt{\frac{1+v}{1-v}}\right)^n < 1, \tag{3.259}$$

giving

$$n < \frac{\ln\left(\frac{2a}{b}\right)}{\ln\left(\sqrt{\frac{1+v}{1-v}}\right)}. \tag{3.260}$$

For given a, b and v let N be the largest integer satisfying (3.260). Then for generic f, the solution Φ will not be $N+1$ times differentiable, i.e. Φ will only have finite differentiability and will not be smooth.

3.3.1 Initial data posed on a global spacelike hypersurface proceeding the Cauchy horizon

Here we look at giving initial data for the Cauchy problem on a global spacelike hypersurface preceding the chronology horizon. Normally (in the absence of wormholes) one would pose initial data for Φ and its normal derivative on a global spacelike hypersurface, and this would be sufficient to determine the unique evolution of Φ. Does the presence of CTCs and the wormholes alter this? The answer would appear to be no. One might worry that the presence of the wormhole (even in the region without CTCs) might make it difficult to find suitable surfaces on which to pose initial data. One can find smooth spacelike surfaces that pass smoothly through both wormhole mouths. One can then evolve the data forward as a sum of components that have traversed the wormhole in a specific sequence (compare with the last section). Friedman et al. speculated that it is reasonable to expect this sum to be a convergent power series in $b/2a$ as before. Once again the diverging lens property of the wormhole prevents any instabilities.

3.3.2 Initial data posed in the region with CTCs

The problem of posing initial data in the region of the spacetime with CTC's is made complicated by the fact that there are no suitable global spacelike hypersurfaces on which to pose our data. Any spacelike surface in this region will inevitably encounter mouth 1 of the wormhole, and subsequently emerge from mouth 2 at some point in the future. It will again then encounter mouth 1 and emerge from mouth 2 still further in the future. Thus a global spacelike hypersurface in the region with CTCs has an infinite number of sheets. A surface like this is not a suitable one on which to pose initial data. There are no global spacelike hypersurfaces in the non-chronal region that smoothly thread both the wormhole mouths. It seems that the difficulties encountered for the Cauchy problem in the region with CTCs are due to the fact that there are not any suitable Cauchy surfaces, as opposed to the actual CTCs themselves. It is also apparently that the problem is a global, and not local one.

So to sum up, it was shown that the diverging lens property of the wormholes seems to stabilise the chronology horizon, and that this same mechanism also seems to ensure a well defined Cauchy problem for the wormhole spacetime given initial data on

$\mathcal{I}^-$, or any global spacelike hypersurface preceding the chronology horizon. However Friedman et al. only proved their results in the geometric optics limit assuming wave-like solutions to the wave equation. Although the results seem convincing on physical grounds it is difficult to prove them using rigorous mathematics.

Chapter 4

THE SPINNING COSMIC STRING

Another spacetime that contains CTCs is that of the spinning cosmic string [DJ]. The spinning cosmic string spacetime $(M_{sc}, \mathbf{g}'')$ has the following metric

$$ds^2 = (dt + 4Jd\theta)^2 - dr^2 - (1-4m)^2 r^2 d\theta^2 - dz^2. \tag{4.1}$$

These are idealised cosmic strings of infinite length and arbitrary thinness. In (4.1) J is the intrinsic spin per unit length of the string, and m is the mass per unit length of the string. There is no structure in the z-direction, and $0 \leq \theta \leq 2\pi$. The spacetime admits closed time like curves, and for sufficiently small radius

$$r < \frac{A}{B}, \tag{4.2}$$

where $A = 4J$ and $B = (1-4m)$, the Killing vector $\frac{\partial}{\partial\theta}$ becomes timelike. We will call the region where $r < \frac{A}{B}$ the *elliptic* region, and the region where $r > \frac{A}{B}$ the *hyperbolic* region for reasons that will become obvious later. The surface $r = \frac{A}{B}$ will be referred to as the 0^{th} polarised hypersurface, again for reasons that will become obvious later. As the structure is uniform in the $z-$direction we will only work in the three-dimensional spacetime $(M_{sp}, \mathbf{g})$ with metric

$$ds^2 = dt^2 - dr^2 + (A^2 - B^2r^2)d\theta^2 + 2Adtd\theta. \tag{4.3}$$

We will now look at the local behaviour of the lightcone in the spinning string spacetime. To help us understand this we first consider vectors lying in the $r = constant$ plane:

$$\mathbf{V} = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \theta}. \tag{4.4}$$

This vector lies in the $r = constant$ plane. The "length" of this vector is given by

$$g(\mathbf{V},\mathbf{V}) = \alpha^2 g_{tt} + \beta^2 g_{\theta\theta} + 2\alpha\beta g_{t\theta}. \tag{4.5}$$

Now $\mathbf{V}$ is null if

$$\alpha^2 + (A^2 - B^2r^2)\beta^2 + 2A\alpha\beta = 0. \tag{4.6}$$

Giving

$$\alpha = (-A \pm Br)\beta. \tag{4.7}$$

For $r > 0$ we have two distinct roots real roots of $g(\mathbf{V},\mathbf{V}) = 0$, thus giving two linear combinations of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \theta}$ that are null. These are given by

$$\mathbf{V}^+ = (-A + Br)\beta\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \theta} \tag{4.8}$$

$$\mathbf{V}^- = (-A - Br)\beta\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \theta}. \tag{4.9}$$

Thus the light cone will cut the surface $r = constant$ twice. Notice that $\mathbf{V}^+$ changes sign as we cross the 0^{th}-polarised hypersurface.

We next consider a vector that in the $t = constant$ plane.

$$\mathbf{W} = \gamma\frac{\partial}{\partial r} + \delta\frac{\partial}{\partial \theta}. \tag{4.10}$$

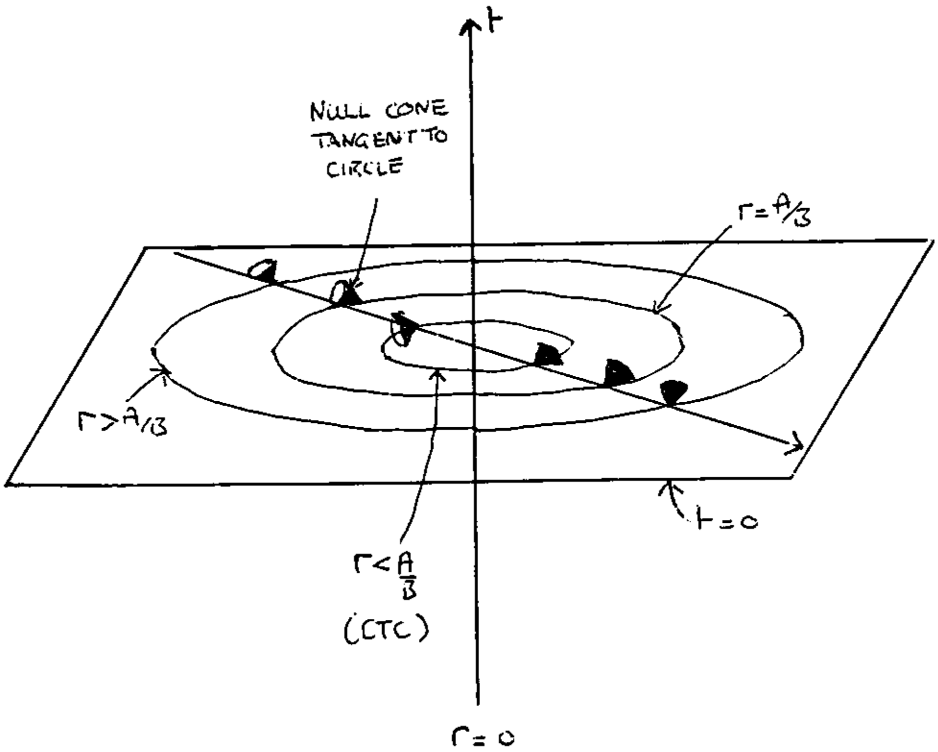

Figure 4.1: *The spinning cosmic string spacetime, with the irrelevant z coordinate suppressed. As r decreases the light cones tip over and open out resulting in the closed timelike curves.*

W is null when

$$\gamma^2 = \delta^2(A^2 - B^2r^2). \tag{4.11}$$

So for $A^2 - B^2r^2 > 0$ there are two distinct linear combinations of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ that are null. For $A^2 - B^2r^2 = 0$ there is only one linear combination resulting in a null vector, that being $W = \delta\frac{\partial}{\partial \theta}$. If $A^2 - B^2r^2 < 0$ then no linear combination of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ will result in a null vector. Thus in the elliptic region the lightcone will intersect the surface $t = constant$ twice. In figure (4.1) one can see how the light-cones tip over as r increases.

4.0.3 Killing Vectors, Conserved Quantities and Geodesics

The spinning cosmic string spacetime has two Killing vectors; a timelike translation Killing vector $\mathbf{t} = \frac{\partial}{\partial t}$ and a rotation Killing vector $\mathbf{m} = \frac{\partial}{\partial \phi}$. Each of these Killing vectors gives rise to quantities that are conserved along geodesics. For the timelike Killing vector, t, we have the quantity E. For timelike geodesics E can be interpreted as representing the total energy per rest mass of a particle following the geodesic in question as measured by an observer at infinity. In the null case hE represent the total energy of a photon. E is given by

$$\begin{aligned} E &= g_{ab} t^a \frac{dx^b}{d\lambda} \\ &= \frac{dt}{d\lambda} + A\frac{d\phi}{d\lambda}. \end{aligned} \tag{4.12}$$

The rotational Killing vector m yields the constant of motion J given by

$$\begin{aligned} J &= g_{ab} m^a \frac{dx^b}{d\lambda} \\ &= A\frac{dt}{d\lambda} + (A^2 - B^2 r^2)\frac{d\phi}{d\lambda} \\ &= AE - B^2 r^2 \frac{d\phi}{d\lambda}. \end{aligned} \tag{4.13}$$

We may interpret J as the angular momentum per unit rest mass of a particle in the timelike case, and we may interpret hJ as the angular momentum of a photon in the null case. In the above λ is some affine parameter along a geodesic. We will use these conserved quantities to enable us to find the equations of the null geodesics.

Consider the metric for the spinning string

$$ds^2 = (dt + Ad\phi)^2 - dr^2 - B^2 r^2 d\phi^2. \tag{4.14}$$

Substituting the (4.12) and (4.13) into (4.14)

$$k^2 = E^2 - \left(\frac{dr}{d\lambda}\right)^2 - \frac{(AE-J)^2}{B^2r^2}, \tag{4.15}$$

where $k = 0,\ 1$ for null and timelike geodesics respectively. For timelike geodesics λ is the proper time, and for null geodesics λ is an affine parameter along the geodesic. Expanding (4.15) we get

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{B^2r^2 - A^2}{B^2r^2}\right)\left(E^2 + \frac{2AJ}{B^2r^2 - A^2}E - \frac{J^2 + k^2B^2r^2}{B^2r^2 - A^2}\right). \tag{4.16}$$

Equation (4.16) gives the radial behaviour of the geodesics. We can simplify (4.16) to give

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{(B^2r^2 - A^2)}{B^2r^2}(E - V^+_{eff})(E - V^-_{eff}), \tag{4.17}$$

where

$$V^+_{eff} = -\frac{AJ}{B^2r^2 - A^2} + \sqrt{\frac{J^2B^2r^2}{(B^2r^2 - A^2)^2} + \frac{k^2B^2r^2}{B^2r^2 - A^2}}, \tag{4.18}$$

$$V^-_{eff} = -\frac{AJ}{B^2r^2 - A^2} - \sqrt{\frac{J^2B^2r^2}{(B^2r^2 - A^2)^2} + \frac{k^2B^2r^2}{B^2r^2 - A^2}}. \tag{4.19}$$

So we can see that the equation governing the radial behaviour of the geodesics reduces to an equation describing the motion of a particle in the presence of an "effective potential" $V^{\pm}_{eff}$.

In the null case $k = 0$; (4.18) and (4.19) become:

$$V^+_{eff} = \frac{J}{A + Br}, \tag{4.20}$$

$$V^-_{eff} = \frac{J}{A - Br}. \tag{4.21}$$

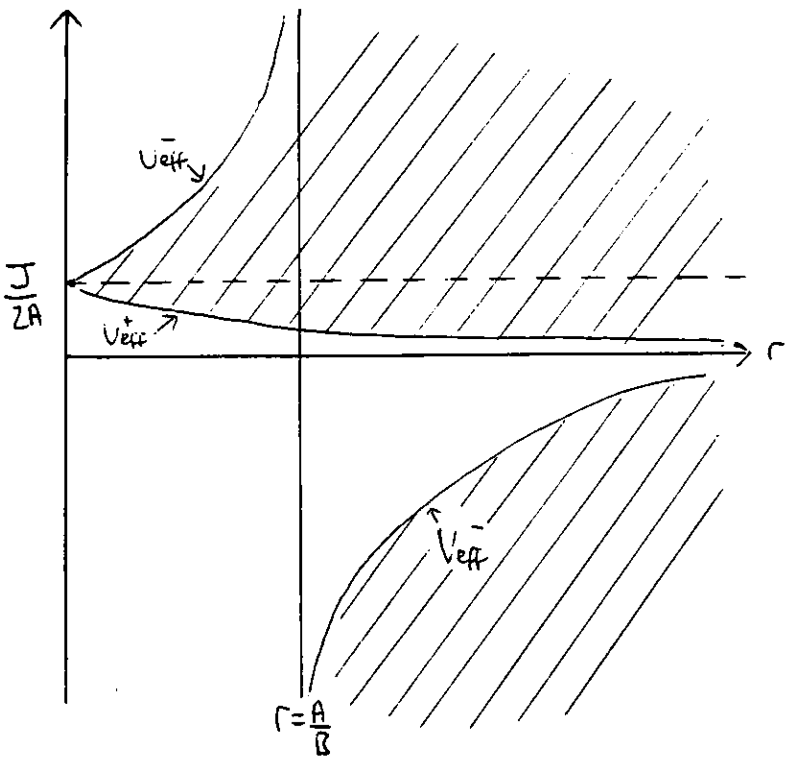

Figure 4.2: *The effective potentials V^+_{eff} and V^-_{eff}.*

Let us consider the radial behaviour of the null geodesics. First consider the case when $J = 0$. For $J = 0$ we get

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{(B^2r^2 - A^2)}{B^2r^2}E^2. \tag{4.22}$$

We see that in this case, a photon following this trajectory cannot enter the 0^{th}-polarised hypersurface, i.e. it cannot obtain values of $r < \frac{A}{B}$.

Now consider the case where $J > 0$. In this case the analysis breaks down on the 0^{th}-polarised hypersurface where V^-_{eff} diverges. However, the analysis still holds arbitrarily close to this surface. The effective potential can be seen in figure (4.2). Now let us consider the regions the geodesics can occupy. For $r > \frac{A}{B}$ we must have

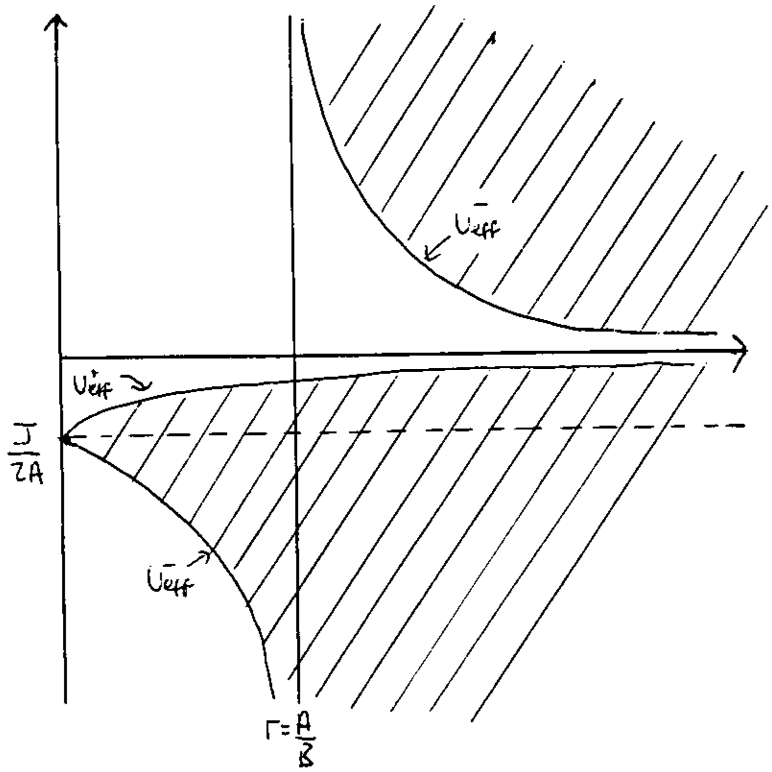

Figure 4.3: *The effective potentials in the case $J < 0$.*

$E > V_{eff}^{+}$ and $E > V_{eff}^{-}$, or $E < V_{eff}^{+}$ and $E < V_{eff}^{-}$. For $r < \frac{A}{B}$ we must have $E < V_{eff}^{+}$ and $E > v_{eff}^{-}$. The allowed regions for the geodesics are the shaded regions of figure (4.2), we can see that photons with $E < \frac{J}{2A}$ are unable to reach the 0^{th}-polarised hypersurface. For $E > \frac{J}{2A}$ the photons can enter the 0^{th}-polarised hypersurface. Only photons with $E = \frac{J}{A}$ are able to reach the string itself. As the energy of the photon gets higher, it can get less and less further into the elliptic region.

A similar analysis can be done for $J < 0$, where the effective potential can be seen in figure (4.3)

Now although the above analysis breaks down on $r = \frac{A}{B}$ it still holds arbitrarily close to that surface. For $J > 0$ photons with $E < 0$ following null geodesics appear to encounter an increasing potential barrier as they approach $r = \frac{A}{B}$. Similarly for $J < 0$ photons with $E > 0$ encounter an increasing potential barrier as the approach $r = \frac{A}{B}$. For $J = 0$ photons cannot enter the elliptic region. We can see that a photon must have non-zero angular momentum in order to enter the elliptic region.

Now consider what happens on $r = \frac{A}{B}$. We get

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{J}{A^2}(2AE - J). \tag{4.23}$$

For $J > 0$ a photon can exist at $r = \frac{A}{B}$ only for values of $E > \frac{J}{2A}$. Similarly for $J < 0$ a photon can exist at $r = \frac{A}{B}$ for $E < \frac{J}{2A}$ only. When $J = 0$ we get

$$\left(\frac{dr}{d\lambda}\right)^2 = 0. \tag{4.24}$$

So we have a turning point for r here. We can deduce that r obtains a minimum value at $r = \frac{A}{B}$ for $J = 0$, and this is consistent with the previous analysis that showed a photon could not enter the elliptic region if we had $J = 0$.

We can in fact find explicit formula for the geodesics of the spinning cosmic string by solving the geodesic equation

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0, \tag{4.25}$$

where s is some affine parameter along the geodesic. In what follows all the k_{ij} are constants. For the spinning string the Christoffel symbols are given by

$$\Gamma^t_{r\theta} = -\frac{A}{r},\ \Gamma^\theta_{r\theta} = \frac{1}{r},\ \Gamma^r_{\theta\theta} = -B^2 r. \tag{4.26}$$

We then get

$$\frac{d^2t}{ds^2} - \frac{2A}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0, \tag{4.27}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0, \tag{4.28}$$

$$\frac{d^2r}{ds^2} - B^2r\left(\frac{d\theta}{ds}\right)^2 = 0. \tag{4.29}$$

Equation (4.28) gives us

$$\frac{d\theta}{ds} = \frac{k_{11}}{r^2}. \tag{4.30}$$

Putting this into (4.29) gives us

$$\frac{d^2r}{ds^2} - \frac{B^2k_{11}}{r^3} = 0. \tag{4.31}$$

This can be solved, giving

$$r^2 = \frac{k_{12}^2(s-s_0)^2 + B^2k_{11}^2}{k_{12}}. \tag{4.32}$$

Using (4.32) we can now solve (4.27) and (4.28) to get

$$(s - s_0) = \pm\frac{Bk_{11}}{k_{12}}\tan B(\theta - \theta_0), \tag{4.33}$$

$$(t - t_0) + A(\theta - \theta_0) = k_{13}(s - s_0). \tag{4.34}$$

So putting $r_0 = \frac{B^2k_{11}^2}{k_{12}}$ the geodesic equations become:

$$r^2 - r_0^2 = k_{12}(s - s_0)^2, \tag{4.35}$$

$$r_0^2\tan^2 B(\theta - \theta_0) = k_{12}(s - s_0)^2, \tag{4.36}$$

$$((t - t_0) + A(\theta - \theta_0))^2 = k_{13}^2(s - s_0)^2. \tag{4.37}$$

Now consider null geodesics, $ds^2 = 0$:

$$\left(\frac{dt}{ds} + A\frac{d\theta}{ds}\right)^2 - \left(\frac{dr}{ds}\right)^2 - B^2r^2\left(\frac{d\theta}{ds}\right)^2 = 0. \tag{4.38}$$

(4.38) gives us

$$k_{13}^2\frac{k_{12}^2}{r^2}(s-s_0)^2 - B^2\frac{k_{11}^2}{r^2} = 0. \tag{4.39}$$

Putting $B^2k_{11}^2 = k_{12}r_0$ and $(s-s_0)^2 = k_{12}(r^2 - r_0^2)$ in (4.39) we get $k_{13}^2 = k_{12}$ for null geodesics.

So the geodesic equations now become:

$$\begin{aligned} r^2 - r_0^2 &= k_{13}^2(s-s_0)^2, & (4.40)\\ r_0^2\tan^2 B(\theta-\theta_0) &= k_{13}^2(s-s_0)^2, & (4.41)\\ ((t-t_0) + A(\theta-\theta_0))^2 &= k_{13}^2(s-s_0)^2, & (4.42)\end{aligned}$$

Taking the positive square root in (4.42) (so that as s goes to infinity we have a positive coordinate time) and writing t, r and θ in terms of s we get:

$$\begin{aligned} t - t_0 &= \left(k_{13}(s-s_0) \mp \frac{A}{B}\tan^{-1}\left(\frac{k_{13}}{r_0}(s-s_0)\right)\right), & (4.43)\\ r^2 - r_0^2 &= k_{13}^2(s-s_0)^2, & (4.44)\\ \theta - \theta_0 &= \pm\frac{1}{B}\tan^{-1}\left(\frac{k_{13}}{r_0}(s-s_0)\right). & (4.45)\end{aligned}$$

We can eliminate s, giving:

$$\begin{aligned} r^2 &= r_0^2\sec^2 B(\theta-\theta_0), & (4.46)\\ ((t-t_0) + A(\theta-\theta_0))^2 &= r_0^2\tan^2 B(\theta-\theta_0), & (4.47)\\ t - t_0 &= \left(\sqrt{r^2-r_0^2} \mp \frac{A}{B}\tan^{-1}\sqrt{\frac{r^2-r_0^2}{r_0^2}}\right). & (4.48)\end{aligned}$$

In (4.46), (4.47) and (4.48) t_0, r_0 and θ_0 are the value of t, r and θ respectively when $s = s_0$. r_0 is the minimum value of r along a given geodesic.

We now want to look at the behaviour of the null geodesics in the spinning string spacetime. Let us consider the how t, the coordinate time, changes with the affine parameter s along the geodesic. Consider first geodesics in the t direction given by

$$t - t_0 = k_{13}(s - s_0) - \frac{A}{B}\tan^{-1}\left(\frac{k_{13}}{r_0}(s - s_0)\right). \tag{4.49}$$

These are future directed geodesics. The θ variation of these geodesics is in the direction of the strings rotation. Differentiating t with respect to s we get

$$\frac{dt}{ds} = k_{13} - \frac{Ak_{13}r}{B(r^2 + k_{13}^2(s - s_0)^2))}. \tag{4.50}$$

We can see that t obtains a maximum, or minimum value as s varies if

$$Bk_{13}^2(s - s_0)^2 = Ar_0 - Br_0^2. \tag{4.51}$$

In order to get a turning point for real values of s we must have $r_0 \leq \frac{A}{B}$. Calculating the second derivative of t with respect to s we get

$$\frac{d^2t}{ds^2} = \pm\frac{2Ak_{13}^3 r_0(s - s_0)}{B(r^2 + k_{13}^2(s - s_0)^2)^2}. \tag{4.52}$$

We can see that for $r_0 = \frac{A}{B}$ we have one turning point, which is an inflection point at $s = s_0$, or $r = \frac{A}{B}$. For values of $r_0 < \frac{A}{B}$ we get two turning points, a maximum and minimum. The maximum and minimum both occur at the same value of r (but different values of θ) given by

$$r^2 = \frac{A}{B}r_0 < \left(\frac{A}{B}\right)^2, \tag{4.53}$$

where s is equal to

$$s = s_0 \pm \frac{k_{13}}{\sqrt{B}}\sqrt{Ar_o - Br_0^2}. \tag{4.54}$$

So we see that if the impact parameter r_0 of the null geodesic is less that $\frac{A}{B}$ then at some point the coordinate time along the geodesic will run in an opposite direction to the affine parameter s.

Now consider geodesics where

$$t - t_0 = k_{13}(s - s_0) + \frac{A}{B}\tan^{-1}\left(\frac{k_{13}}{r_0}(s - s_0)\right). \tag{4.55}$$

These correspond to geodesics moving against the rotation of the string. Differentiating with respect to s we get

$$\frac{dt}{ds} = k_{13}\left(1 + \frac{Ar_0}{B(r_0^2 + k_{13}^2(s - s_0)^2)}\right). \tag{4.56}$$

Here we have no solutions of $\frac{dt}{ds} = 0$.

From the above we see that the coordinate time of null geodesics that move with the rotation of the string obtain local maxima and minima with respect to the affine parameter s if they enter the 0^{th}-polarised hypersurface. For geodesics moving against the spin of the string the coordinate time t is an always increasing function of s. The behaviour of the null geodesics that move in the direction of the strings rotation is illustrated in figures (4.4), (4.5) and (4.6).

Now consider the angular behaviour of null geodesics. We have

$$\theta - \theta_0 = \pm\frac{1}{B}\tan^{-1}\left(\frac{k_{13}}{r_0}(s - s_0)\right).$$

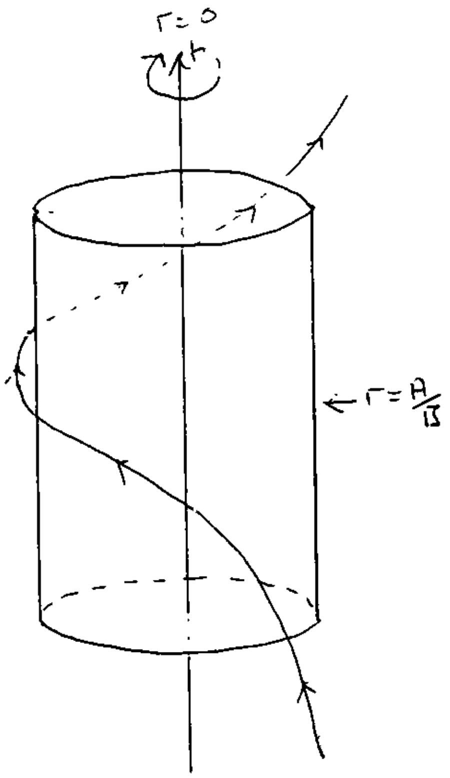

Figure 4.4: *Null geodesics in the spinning cosmic string spacetime, with* $r_0 > \frac{A}{B}$.

As s varies from $s = -\infty$ to $s = \infty$ the value of θ will vary as follows:

$$-\frac{\pi}{2B} < \theta - \theta_0 < \frac{\pi}{2B}. \tag{4.57}$$

Only for $0 < B < \frac{1}{2}$, when the deficit angle of the string is between π and 2π, can geodesics wind round the string n times, where $n \geq 1$. In fact the winding number of a geodesic is the integer part of $\frac{1}{2B}$. Obviously this means that we can only have self intersecting null geodesics in the spinning string spacetime when $0 < B < \frac{1}{2}$.

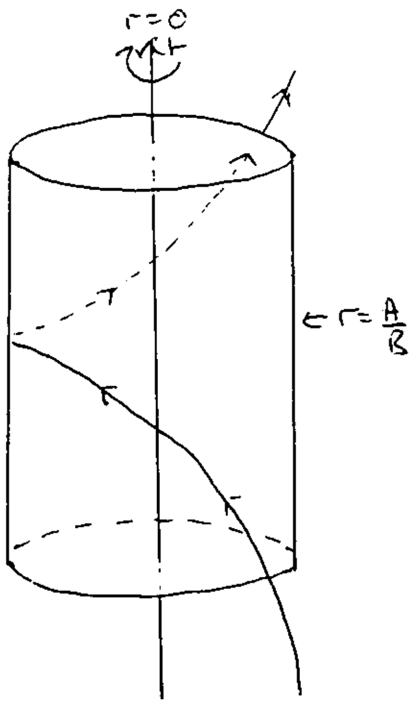

Figure 4.5: *Null geodesics in the spinning cosmic string spacetime, with* $r_0 = \frac{A}{B}$.

Polarised hypersurfaces.

In order to have a self intersecting null geodesic that winds round the string n times we require that the values of t and r have the same value at θ and $\theta + 2n\pi$. As $t(\theta) = t(\theta + 2n\pi)$ the value of t will obtain a local minimum/maximum , thus we need only consider geodesics moving in the direction of the strings rotation. For a null geodesic $t(\theta)$ and $r(\theta)$ are given by

$$t - t_0 = -A(\theta - \theta_0) + r_0 \tan B(\theta - \theta_0) \tag{4.58}$$

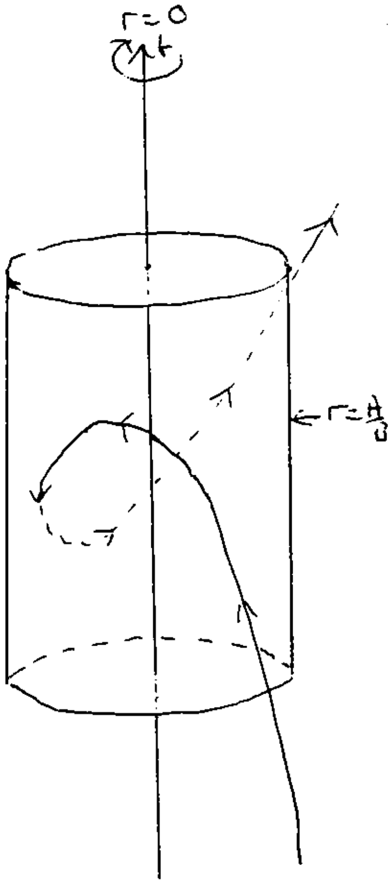

Figure 4.6: ***Null geodesics in the spinning cosmic string spacetime, with*** $r_0 > \frac{A}{B}$.

$$r \quad = \quad r_0 \sec B(\theta - \theta_0). \tag{4.59}$$

Without loss of generality take $t_0 = 0$ and $\theta_0 = 0$. Now we require

$$r(\theta) = r_0 \sec B(\theta) = r_0 \sec B(\theta + 2n\pi) = r(\theta + 2n\pi). \tag{4.60}$$

To satisfy (4.60) we must have

$$\theta = -n\pi + \frac{m\pi}{B},$$

where $m \in \mathbf{Z}$. At these value of θ

$$t(\theta) = An\pi - \frac{Am\pi}{B} - r_0 \tan Bn\pi,$$

and

$$t(\theta + 2n\pi) = -An\pi - \frac{Am\pi}{B} + r_0 \tan Bn\pi.$$

For $t(\theta)$ to equal $t(\theta + 2n\pi)$ we must have

$$r_0 = \frac{An\pi}{\tan Bn\pi} < \frac{A}{B}.$$

Putting this value of r_0 back into (4.59) we find the n^{th}-polarised hypersurfaces are located at

$$r = \frac{An\pi}{\sin Bn\pi},$$

where

$$n \leq \left|\frac{1}{2B}\right|.$$

4.0.4 The Covering Space

We may locally transfer the spinning cosmic string metric into that of the Minkowski metric by introducing new coordinates:

$$\tau = t + A\theta, \ \phi = B\theta, \tag{4.61}$$

giving

$$ds^2 = d\tau^2 - dr^2 - r^2 d\phi^2, \tag{4.62}$$

where $0 \leq \phi \leq 2\pi B$. The spinning string spacetime $(M_{sc}, \mathbf{g}'')$ is the quotient of Minkowski space $(M, \mathbf{g})$ under the group G_{sc}. If E generates G_{sc}, then E^n maps the point (τ, r, ϕ) to the point

$$(\tau + 2nA\pi, r, \phi + 2nB\pi). \tag{4.63}$$

So we see that the spinning cosmic string spacetime can be thought of as Minkowski space with points identified under a translation in time and a rotation in the $x - y$ plane. One has to be careful when putting these identifications in terms of Cartesian coordinates. Naively one might think we could put

$$x = r\cos\phi, \;\; y = r\sin\phi,$$

resulting in the point (τ, x, y) being identified with

$$(\tau + 2An\pi, x\cos 2Bn\pi - y\sin 2Bn\pi, x\sin 2Bn\pi + y\cos 2Bn\pi). \tag{4.64}$$

In Cartesian coordinates the real isometries of the spacetime become distorted because of the periodic nature of cos and sin used in the definition of x and y. To illustrate this consider the case $n = 1$, i.e. a null geodesic that circles the string once before self intersecting. According to (4.64) the point (t, x, y) is identified with the point

$$(\tau + 2A\pi, x\cos 2B\pi - y\sin 2B\pi, x\sin 2B\pi + y\cos 2B\pi).$$

However this is only true if one goes in the positive ϕ direction (recall (τ, r, ϕ) is identified with $(t + 2A\pi, r, \phi + 2B\pi)$). To see this consider figures (4.7) and (4.8), P_0

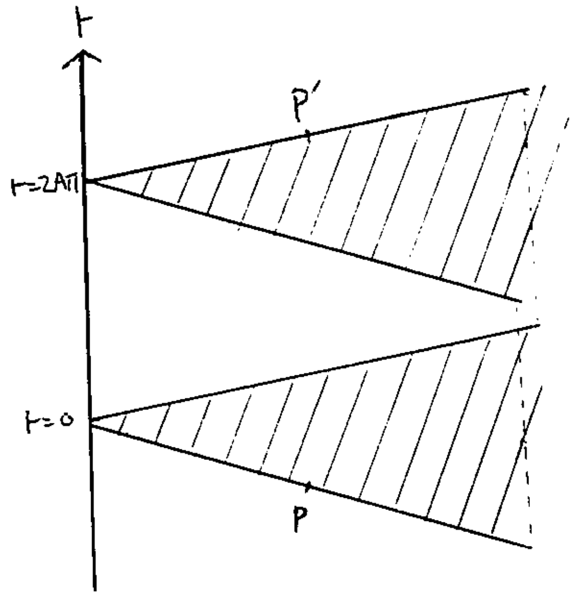

Figure 4.7: *The Spinning Spacetime with deficit angle between* 0 *and* π

is identified with P_1 going via A but not B. The coordinates (τ, x, y), where

$$x = r\cos\phi,\ \ y = r\sin\phi,$$

are only valid for $0 \leq \phi \leq 2B\pi$. We need to define new coordinates to cover the spinning string spacetime for $\phi > 2B\pi$.

We can see from figure (4.7) that there exists no straight lines joining P to its image P' in the direction of increasing ϕ in the covering space in the case $\frac{1}{2} < B < 1$ (deficit angle between 0 and π). Therefore there are no geodesics joining P to P' in the physical space. We therefore have no polarised hypersurfaces in the spinning string spacetime for $\frac{1}{2} < B < 1$.

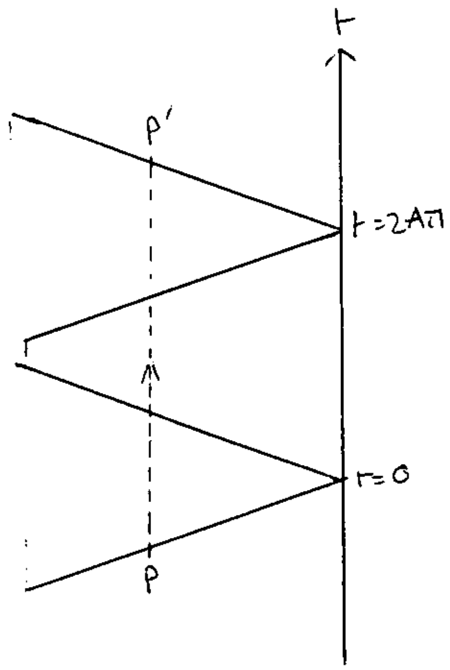

Figure 4.8: *The Spinning Spacetime with deficit angle between* π *and* 2π

However consider figure (4.8), one can see that in the case where $0 < B < \frac{1}{2}$ (deficit angle between π and 2π) it is possible to join P to P' by a straight line in the covering space, and therefore by a geodesic in the physical space. In fact we see that we can join P to its n^{th} image point by a geodesic where

$$n < \left|\frac{1}{B}\right|. \tag{4.65}$$

The polarised hypersurfaces are located at

$$r^2 = \frac{An\pi}{\sin Bn\pi}, \tag{4.66}$$

where the values of n are given by (4.65). The larger the deficit angle the more polarised hypersurfaces we have. Taking the $n \to 0$ limit of (4.66) gives us the "0^{th}-

polarised hypersurface" $r = \frac{A}{B}$. We see that the results obtained in the covering space are consistent to those obtained before.

4.0.5 The Wave Equation

Let us now consider the wave equation on the spinning cosmic string spacetime.

$$\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\Phi\right) = 0. \tag{4.67}$$

The wave equation becomes:

$$\frac{B^2r^2 - A^2}{Br}\partial_t^2\Phi - Br\partial_r^2\Phi - B\partial_r\Phi - \frac{1}{Br}\partial_\theta^2\Phi + \frac{2A}{Br}\partial_t\partial_\theta\Phi = 0. \tag{4.68}$$

As θ is a periodic coordinate we can Fourier analyse in θ. A general solution Φ to the full wave equation can be written as a finite sum of the modal solutions Ψ_k that satisfy:

$$\Phi(t,r,\theta) = \Re\left(\sum_k e^{ik\theta}\Psi_k(t,r),\right) \tag{4.69}$$

where k is an integer. We get:

$$\frac{B^2r^2 - A^2}{Br}\partial_t^2\Psi_k - Br\partial_r^2\Psi_k - B\partial_r\Psi_k + \frac{2Aik}{Br}\partial_t\Psi_k + \frac{k^2}{Br}\Psi_k = 0. \tag{4.70}$$

We will call equation (4.70) the reduced wave equation. The reduced wave equation changes type over the 0^{th}-polarised hypersurface.

For $r < \frac{A}{B}$ the equation is elliptic, and for $r > \frac{A}{B}$ the equation is hyperbolic. As the reduced wave equation becomes elliptic for $r < \frac{A}{B}$ this raises the question of whether or not the full wave equation has a well posed Cauchy problem over the whole spacetime. We will call the region $r < \frac{A}{B}$ the elliptic region and the region $r > \frac{A}{B}$ the hyperbolic region. The 0^{th}-polarised hypersurface is the surface of parabolic degeneracy.

Let us consider the characteristics of the reduced wave equation. These are determined by the following equations

$$\pm \frac{\sqrt{B^2r^2 - A^2}}{Br} dr = dt, \tag{4.71}$$

with solution

$$\pm t + \sqrt{r^2 - \frac{A^2}{B^2}} - \frac{A}{B} \tan^{-1}\left(\frac{\sqrt{B^2r^2 - A^2}}{A}\right) = C, \tag{4.72}$$

where C is an arbitrary function.

Notice that characteristics only exist in the hyperbolic region. Pairs of characteristics intersect, forming cusps on the surface $r = \frac{A}{B}$.

For simplicity let us first consider solving the reduced wave equation for $k = 0$. Note that solutions of this equation are also the θ independent solutions of the full wave equation. With $k = 0$ the reduced wave equation becomes:

$$(B^2r^2 - A^2)\partial_t^2\Psi - B^2r^2\partial_r^2\Psi - B^2r\partial_r\Psi = 0. \tag{4.73}$$

This can be solved by separation of variables. Put $\Psi(t, r) = T(t)R(r)$, and divide both sides by $B^2r^2 - A^2$. So for $r \neq \frac{A}{B}$ we get:

$$\frac{\partial_t^2 T(t)}{T(t)} = \frac{B^2}{R(r)(B^2r^2 - A^2)}\left(r^2\partial_r^2 R(r) + r\partial_r R(r)\right) = M, \tag{4.74}$$

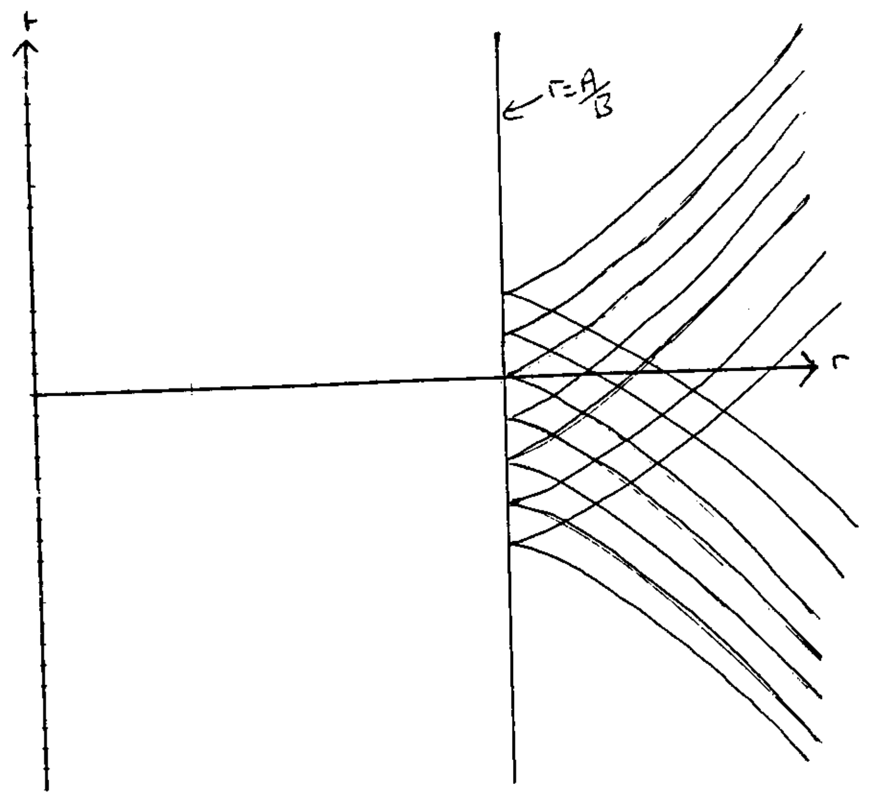

Figure 4.9: *The characteristics for the Fourier analysed wave equation in the spinning cosmic string spacetime.*

where M is the separation constant. We now consider different values of the separation constant. In what follows the k_m are all constants.

Case(i) $M < 0$

Putting $M = -w^2$ the t part of the equation can be solved to give

$$T(t) = k_1 e^{iwt} + k_2 e^{-iwt}. \tag{4.75}$$

The r part of the equation becomes the Bessel equation having solutions:

$$R(r) = k_3 J_{\frac{wA}{B}}(wr) + k_4 Y_{\frac{wA}{B}}(wr). \tag{4.76}$$

In the above J and Y are Bessel functions of the first and second kind respectively. J converges everywhere, whereas Y becomes infinite as r approaches zero. To get a solution that is finite at the origin we set $k_4 = 0$. So a θ independent solution to the wave equation on the spinning string spacetime is given by:

$$\Phi(t,r) = k_3(k_1 e^{iwt} + k_2 e^{-iwt}) J_{\frac{wA}{B}}(wr). \tag{4.77}$$

Case(ii) $M = 0$

We have

$$\frac{\partial_t^2 T(t)}{T(t)} = \frac{B^2}{R(r)(B^2r^2 - A^2)} \left(r^2\partial_r^2 R(r) + r\partial_r R(r)\right) = 0. \tag{4.78}$$

So the t equation is simply

$$\partial_t^2 T(t) = 0. \tag{4.79}$$

This has solution

$$T(t) = k_5 t + k_6. \tag{4.80}$$

The r equation is

$$r^2\partial_r^2 R + r\partial_r R = 0. \tag{4.81}$$

We solve this by making the substitution $\partial_r R = Z$, which gives

$$r^2\partial_r Z + rZ = 0. \tag{4.82}$$

This has solution

$$Z = \frac{k_7}{r}. \tag{4.83}$$

Giving

$$R(r) = \ln k_8 r^{k_9}. \tag{4.84}$$

So the solution in the case $M = 0$ is

$$\Phi(t, r) = (k_{10}t + k_{11}) \ln k_8 r, \tag{4.85}$$

where $k_{10} = k_5 k_9$ and $k_{11} = k_6 k_9$.

Case(iii) $M > 0$

Putting $M = w'^2 > 0$ the t part of the equation simply gives us

$$T(t) = k_{12} e^{w't} + k_{13} e^{-w't}. \tag{4.86}$$

The r part of the equation gives us a modified Bessel equation of imaginary order, so we get

$$R(r) = k_{14} I_{\frac{iw'A}{B}}(w'r) + k_{15} K_{\frac{iw'A}{B}}(w'r). \tag{4.87}$$

In the above I and K are modified Bessel functions of the first and second kind, respectively. They are regular everywhere except at $r = 0$. The third set of solutions to the θ independent wave equation for the spinning string is given by

$$\Phi(t, r) = (k_{12} e^{w't} + k_{13} e^{-w't})(k_{14} I_{\frac{iw'A}{B}}(w'r) + k_{15} K_{\frac{iw'A}{B}}(w'r)). \tag{4.88}$$

So we see that we have constructed three different solutions to the θ independent wave equation. They are all finite everywhere, except at the origin.

4.0.6 The Region with Closed Timelike Curves

Here we will see that no smooth timelike curve can be closed if it remains fully in the hyperbolic region of the spacetime. Consider a closed curve, parameterized by θ given by

$$\begin{aligned} t &= t(\theta), \\ r &= r(\theta). \end{aligned}$$

Along this trajectory we have

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dt}{d\theta}\right)^2 - \left(\frac{dr}{d\theta}\right)^2 + (A^2 - B^2r^2) + 2A\frac{dt}{d\theta}. \tag{4.89}$$

Now as this is a closed curve, there will be some $\theta = \theta'$ such that $\frac{dt}{d\theta} = 0$. At this point the metric becomes

$$\left(\frac{ds}{d\theta}\right)^2 = -\left(\frac{dr}{d\theta}\right)^2 + (A^2 - B^2r^2). \tag{4.90}$$

Now if $r > \frac{A}{B}$ everywhere along the curve (that is, if the curve remains in the hyperbolic region) then when $\theta = \theta'$ the metric will be negative. However along a timelike curve we cannot have $ds^2 < 0$, so we cannot have a closed timelike curve that remains fully in the hyperbolic region.

4.1 Comparisons between the Spinning Cosmic string and Grant space

Recall from chapter 3 the following metric for Grant space in the non-chronal region

$$ds^2 = -d\tau^2 - \frac{\tau^2}{4}(d\mu - d\nu)^2 - \frac{d^2}{4a^2}(d\mu + d\nu)^2. \tag{4.91}$$

In the above ν is periodic, period $8na$. Now put

$$P = \frac{\mu - \nu}{2}, \; Q = -\frac{\mu + \nu}{2A}. \tag{4.92}$$

Giving

$$ds^2 = -d\tau^2 + \tau^2 dP^2 - dQ^2. \tag{4.93}$$

Now both P and Q are periodic coordinates, period $4na$ and $4nd$ respectively. Now put

$$P = a\phi, \; Q = R + \phi d, \tag{4.94}$$

so that the metric now becomes

$$ds^2 = -d\tau^2 - (d^2 - a^2\tau^2)d\phi^2 - dR^2 - 2ddRd\phi. \tag{4.95}$$

The above is the metric for Grant space in the acausal region, where ϕ is a periodic coordinate, period 4.

Now if in the above metric we put $\phi = i\theta$, $\tau = r$ and $R = it$ we recover the metric for the spinning string. We can now write down the equation of the geodesics in the acausal region of Grant space:

$$R - R_0 = -A(\phi - \phi_0) \pm \tau_0 \tanh B(\phi - \phi_0), \tag{4.96}$$

$$\tau = \tau_0 \operatorname{sech} B(\phi - \phi_0). \tag{4.97}$$

We can see that as we pass many times round Grant space (that is, as $\phi \to \infty$) we approach the chronology horizon located at $\tau = 0$. We can also repeat similar calculations done, in the spinning string example, to find the location of the polarised hypersurfaces.

As a result of the above it is not surprising that Grant space and the spinning cosmic string have quite a few common features. Both spacetimes contain and elliptic region where the periodic coordinate becomes timelike. Each spacetime contains self intersecting null geodesics, with intersection points lying in the hyperbolic region (where the periodic coordinate is spacelike). The elliptic and hyperbolic regions are separated by the "0^{th} polarised hypersurface". In both spacetimes it is necessary for CTCs to enter the elliptic region, thus removing this region we remove all causality violation. The main difference between the two spacetimes is that they are each 'inside out versions' of the other. The spinning string has an elliptic region between $r = 0$ and the 0^{th} polarised hypersurface, and then a hyperbolic region that goes out to infinity. Whereas in Grant space we have a Chronology horizon at $\tau = 0$ and the hyperbolic region extends between this and the 0^{th} polarised hypersurface, and the elliptic region goes out to infinity. We will now go on to investigate further the "0^{th}-polarised hypersurface" in the next chapter.

Chapter 5

CHARACTERISATION OF THE 0^{TH} POLARISED HYPERSURFACE AND CONCLUSIONS.

In this final chapter we will give a formal definition of the the "0^{th}-polarised hypersurface" which will apply to general spacetimes without isometries. We will start by giving a review of various properties of the "0^{th}-polarised hypersurface" in the spinning cosmic string spacetime, Grant space and Gott space all of which do not have compactly generated chronology horizons, and contrast this with the behaviour in spacetimes with compactly generated chronology horizons.

5.0.1 Properties of the 0^{th}-polarised hypersurface in Grant space, Gott space and the spinning string spacetime.

The spinning cosmic string spacetime (for string of deficit angle greater that π) , Grant space and Gott space all contain self intersecting null geodesics (SINGS). A SING with winding number n generates the n^{th}-polarised hypersurfaces. One can write down an expression for the n^{th} polarised hypersurface, for example in Gott space and Grant space the n^{th} polarised hypersurface is given in the covering space by values of t and x satisfying:

$$x^2 - t^2 = \frac{d^2 n^2}{\sinh(a^2 n^2)}. \tag{5.1}$$

Taking the limit as n goes to zero in expression (5.1) yields what we call the 0^{th}-polarised hypersurface. So in the case of Gott space and Grant space this is given by:

$$x^2 - t^2 = \frac{d^2}{a^2}.$$

The $n \to \infty$ limit of (5.1) yields the chronology horizon of Gott space and Grant space. It should be noted that in Gott space and Grant space the chronology horizon itself is in fact not in the causality violating region, this is in contrast with Misner space where the chronology horizon contains a closed null geodesic.

Most of the spacetimes studied in this thesis are locally Minkowski space, but have different global structure. For these spacetimes one can perform a change of coordinates so that all the isometries of the spacetime are due to the identifications by one periodic coordinate. The tangents, $\mathbf{K}$, to the orbits of the of the periodic coordinate are Killing vectors that become null on the 0^{th}-polarised hypersurface. It should be noted that the orbits of the periodic Killing vector are not necessarily geodesics. The 0^{th}-polarised hypersurface is a timelike surface that contains closed null curves generated by the vector $\mathbf{K}$. So for example in the case of the spinning string spacetime the 0^{th}-polarised hypersurface is a timelike cylinder given $r = \frac{A}{B}$.

The motivation for such a coordinate transformation was to allow us to solve the wave equation by Fourier analysing with respect to the periodic variable. The result of this Fourier analysis produces a two dimensional equation, the "reduced wave equation", which is an equation of mixed type. That is, it changes from hyperbolic to elliptic over a surface of parabolic degeneracy. The surface of parabolic degeneracy turns out to be the 0^{th}-polarised hypersurface. The equation is elliptic in the region where the orbits of $\mathbf{K}$ are timelike.

We now consider the region of Gott space, Grant space and the spinning string spacetime which is not chronal but is in the hyperbolic region of the reduced wave equation. Although every point in these regions has a closed time like curve through it such a curve cannot lie totally within the hyperbolic region and be closed and timelike, but must at some point enter the elliptic region. We can therefore remove the elliptic region and obtain a causal spacetime. Also it should be noted that in the spinning string spacetime (and in fact in Gott space and Grant space) one cannot slice the spacetime such that the slices are spacelike everywhere. However if one again removes the elliptic region one can slice the spacetime by spacelike hypersurfaces. So in some sense, to be made precise later, the 0^{th}-polarised hypersurface is the boundary of the maximal causal spacetime. Note however that the resulting causal spacetime is not geodesically complete. Whether it is possible to find an extension of this spacetime which is both causal and geodesically complete is an interesting question.

5.0.2 0^{th} Polarised Hypersurfaces in Spacetime with Compactly Generated Chronology Horizons.

In the above we looked at the properties of the 0^{th} polarised hypersurface in spacetime with non-compactly generated chronology horizons (Grant space and Gott space) or spacetimes without chronology horizons (the spinning cosmic string spacetime). We now consider the analogue of the 0^{th}-polarised hypersurface in spacetimes with compactly generated chronology horizons. Consider first Misner space. Recall that this has metric:

$$ds^2 = 2dX'dT + TdX'^2,$$

where $-\infty < T < \infty$ and $0 < X' < 2\pi$. The point $(T, X' = 0)$ is identified with the point $(T, X' = 2\pi)$. The region $T > 0$ contains CTCs, and is separated from the causal region, $T < 0$, by a chronology horizon at $T = 0$. On the horizon $T = 0$ the orbit of the periodic coordinate X' become null, and then timelike for $T > 0$. Note that in the Grant space, Gott space and the spinning string spacetimes the orbits of the periodic coordinates became null on the 0^{th}-polarised hypersurface, and not the chronology horizon.

Again when we considered solving the wave equation on three dimensional Misner space by Fourier analysing the wave equation with respect to the periodic coordinate we found that the resulting resulting reduced wave equation equation was of mixed type. The wave equation became elliptic in the non-causal region and the surface of parabolic degeneracy was $T = 0$, the chronology horizon.

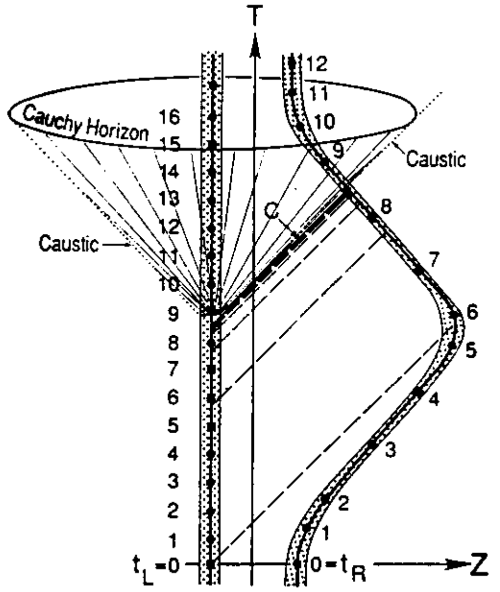

Figure 5.1: *The Twins Paradox Wormhole spacetime.*

In the case of Misner space the orbits of the periodic coordinate X' are again Killing vectors who integral curves are closed geodesics. On the chronology horizon $\frac{\partial}{\partial X'}$ generates a closed null geodesic.

Let us now consider the twins paradox wormholes spacetimes discussed in chapter 2. It is not so easy to form a covering space for this spacetime. So defining the 0^{th}-polarised hypersurface in terms of a periodic coordinate is not straight forward. However if one considers the wormhole throats to be infinitely thin, and considers the wormhole mouths lying in the $t - x$ plane (see fig (5.1). As in Misner space, the chronology horizon is generated by a closed null geodesic.

We therefore find that the 0^{th} polarised hypersurface in Grant space, Gott space and the spinning string spacetime has many of the properties of the chronology horizon in spacetimes with compactly generated chronology horizons.

5.0.3 Possible conjectures for the definition 0^{th}-polarised hypersurface.

So far we have seen that the main feature of the 0^{th}-polarised hypersurface is that it is the surface on which the orbits of the isometries of the spacetime become null, and hence where the Fourier reduced wave equation changes type. As we have seen in the case of the wormhole spacetime, it is not necessarily convenient to describe the 0^{th}-polarised hypersurface in terms of the orbits of the isometries. In fact one might imagine slightly perturbing the cosmic string spacetime so that it no longer had these isometries. We would then need a new way of defining the 0^{th}-polarised hypersurface.

Consider again what happened when we Fourier analysed the wave equation in the various spacetimes. For the moment let us consider the spinning string spacetime. In the hyperbolic region there exists surfaces that are spacelike and achronal with respect to the symbol of the reduced wave equation, but in the elliptic region there exists no such spacelike surfaces. So in the two dimensional Fourier reduced problem the 0^{th}-polarised hypersurface represents a boundary beyond which we cannot find spacelike surfaces on which to pose initial data for a Cauchy problem . In the two dimensional case we cannot extend a surface that is spacelike with respect to the symbol of the reduced wave equation beyond the 0^{th}-polarised hypersurface, and keep it spacelike. This property of the two dimensional problem lead us to conjecture that the 0^{th}-polarised hypersurface separated regions of the spacetime in which it is possible to have smooth surfaces that are spacelike everywhere and extend once round the string and that such surfaces cannot exist in the elliptic region.

This conjecture failed to be true; one can find a counter example. Consider the spinning cosmic string spacetime with metric:

$$ds^2 = dt^2 - dr^2 + (A^2 - B^2r^2)d\theta^2 + 2Adtd\theta.$$

The 0^{th}-polarised hypersurface is located at $r = \frac{A}{B}$. One can find a closed spacelike curve contained totally in the elliptic region, and thicken this out to obtain a spacelike surface. An example of such a curve is given below:

$$t = \frac{\beta}{4\nu}\sin 2\nu\theta, \tag{5.2}$$

$$r = \frac{A}{B} - \epsilon\left(1 + \frac{1}{2}\sin\nu\theta\right). \tag{5.3}$$

This will be spacelike provided:

$$\begin{aligned} 0 < \ &\beta < 2A, \\ 0 < \ &\epsilon < \frac{\beta}{4B}, \\ &\nu > \frac{\sqrt{8A\beta}}{\epsilon}. \end{aligned}$$

However it is not possible to extend this spacelike surface to infinity.

Now consider a spacelike surface in the spinning string spacetime that extends to spacelike infinity, and has an interior boundary ∂S that lies outside the 0^{th}-polarised hypersurface and is such that the boundary curve, ∂S is closed and passes once round the string , see figure (5.2). We conjectured that it would not be possible to extend

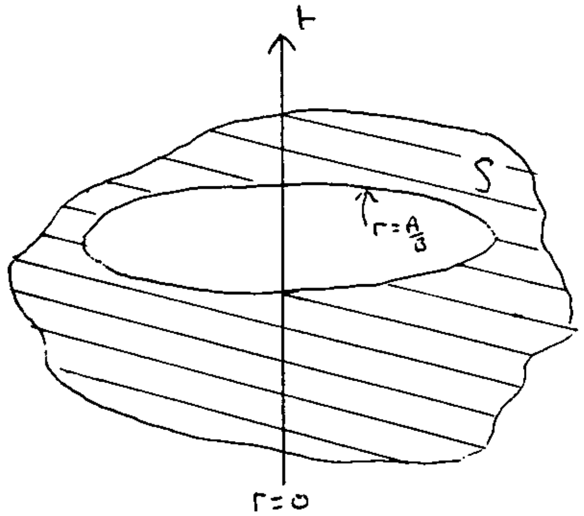

Figure 5.2: *The spacelike surface S.*

the surface ∂S into the elliptic region and keep it spacelike. This however is not true. One could imagine a ∂S such that the intersection of ∂S and the 0^{th}-polarised hypersurface is not the empty set. (Note that if ∂S was the 0^{th}-polarised hypersurface S would not be everywhere spacelike.) Pick a point p that lies on the intersection of ∂S and the 0^{th}-polarised hypersurface, and then pick an open neighbourhood P_0 of p, see figure (5.3). In P_0 the spacetime will be locally Minkowskian, so there will exist a spacelike surface within P_0. S together with its extension P_0 will still be spacelike. However we shall see that such an extension, although spacelike, is not achronal.

We first examine the simpler case of Misner space. Consider a spacelike achronal surface S in the causal region of Misner space, and attempt to extend this surface (as an achronal surface) into the acausal region beyond the chronology horizon. As soon as this surface intersects the chronology horizon it will contain a point p that can be joined to itself by a null geodesic. At this instant S ceases to be achronal. This

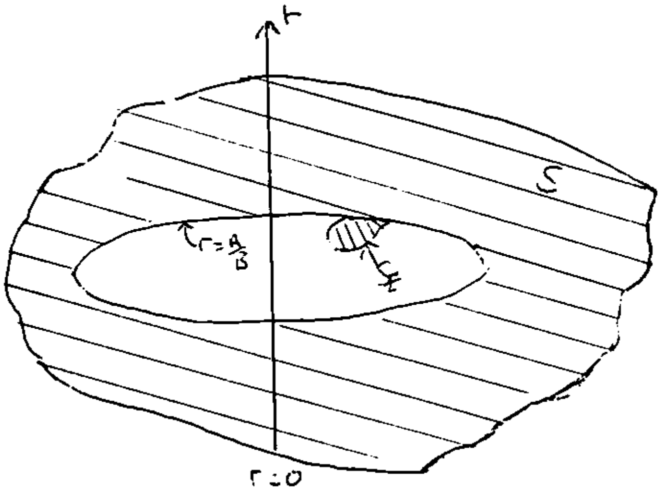

Figure 5.3: *Extending S into the elliptic region.*

then lead us to conjecture that an achronal surface can not be extended across the 0^{th}-polarised hypersurface and remain achronal. This will be put on a more formal basis in the next section.

5.0.4 A general definition of the 0^{th}-polarised hypersurface

In this section we will given a general definition of the 0^{th}-polarised hypersurface. In all the spacetimes discussed the 0^{th}-polarised hypersurface is the surface on which the orbits of the isometries become null. However if one were to perturb such a spacetime it seems likely that the causal structure would remain similar but we would no longer be able no longer be able to define the 0^{th}-polarised hypersurface using either the isometries or reduced wave equation. Indeed, one might well have a causal violating spacetime which is not even close to one with symmetries. We therefore want to find a way of defining the 0^{th}-polarised hypersurface in a way that is not dependent on constructing a reduced wave equation. To motivate the definition we first give a way of defining the 0^{th}-polarised hypersurface in the spinning string spacetime with deficit angle greater than π.

Consider the problem of extending local Cauchy data on a spacelike achronal hypersurface S_L to Cauchy data on a "global" spacelike achronal hypersurface S containing S_L. By a "global" hypersurface we mean one that goes out to spacelike infinity and extends round the string. Pick a point p in the hyperbolic region, and pick an open neighbourhood P of p such that P is fully contained in the hyperbolic region. As the spacetime is locally Minkowski space and hence locally globally hyperbolic we can always find an achronal spacelike surface S_L in P on which we can put data for a Cauchy problem. Now consider extending S_L to a "global" surface S in such a way that we still have a well defined Cauchy problem given initial data on S. It certainly seems likely this can be done in the hyperbolic region. For example is we choose S_L to be part of the hypersurface $t = 0$ then we may choose S to be given by $t = 0$ for $r > \frac{A}{B}$. This is the surface we looked at when we considered the two-dimensional Fourier reduced problem for the spinning string. We saw that given initial on S the reduced wave equation have a unique solution in $D_2^+(S)$, see figure (5.4) There must, therefore, also be a unique solution to the three-dimensional wave equation given data on S_0. However, although we have a well defined Cauchy problem given initial on S, S is in fact not achronal. There will be points p in S that lie on self intersecting null geodesics, and so there are points in S that can be joined to themselves by null geodesics. However S is achronal within its own domain of dependence. In order for

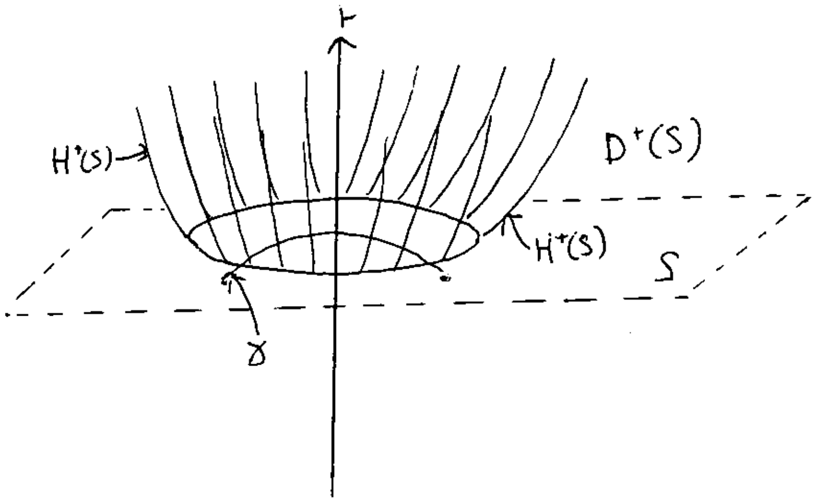

Figure 5.4: *The surface S and its domain of dependence (the region outside the null surface $H^+(s)$. γ is a null geodesic joining a point in S to another point in S.*

a null geodesic to intersect S twice it must enter the elliptic region, and so go outside $D^+(S)$.

Now consider extending the surface S through the 0^{th}-polarised hypersurface. In the elliptic region the surface $t = 0$ becomes timelike with respect to the symbol of the reduced wave equation, and hence the Cauchy problem cannot be extended into the elliptic region. How does this relate to what happens in three-dimensional picture. As we extend the surface S, keeping it spacelike, into the elliptic region S no longer remains achronal in its domain of dependence. Let S_E denote that part of S inside the elliptic region. There will be null geodesics that join points in S_E to points in S while remaining in $D^+(S \cup S_E)$, see figure (5.5).

Definition 5.1 *Let A be the set of all hypersurfaces S such that S has the property*

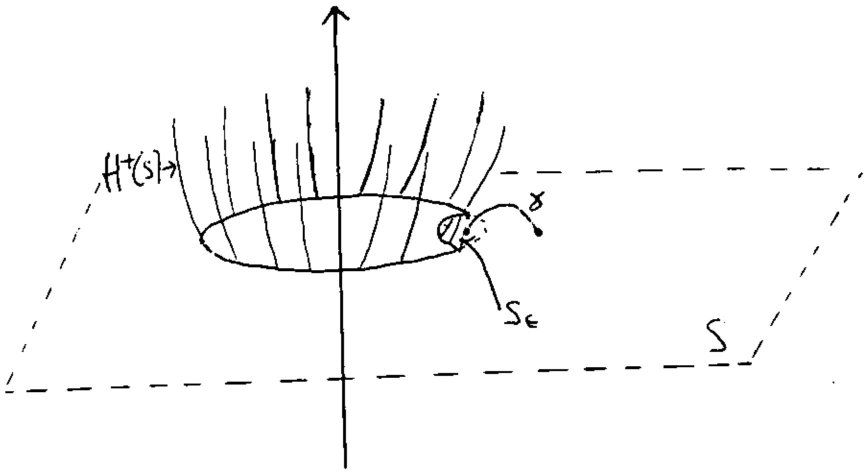

Figure 5.5: *The surface $S \cup S_E$ and its domain of dependence. γ is a geodesic joining a point in S_E to S.*

that it is achronal within $D^+(S)$, where $D^+(S)$ is the domain of dependence of S.

$$A = \{S : S \textit{ is achronal within } D^+(s)\}.$$

Definition 5.2 *A hypersurface $S \in A$ is said to be extendible if there exists a hypersurface $\tilde{S} \in A$ such that $S \subset \tilde{S}$ and $\tilde{S} - S \neq \emptyset$.*

Definition 5.3 *A hypersurface S is said to be maximal if it is not extendible.*

Definition 5.4 *Let B be the set of all maximal surfaces S, and let C be the set of points x such that*

$$C = \{x : x \in S \in B\}.$$

Definition 5.5 *The essential chronology horizon is ∂C, the boundary of the set C*

Conjecture 5.6 *The essential chronology horizon coincides with the 0^{th}-polarised hypersurface of Gott space, Grant space and the spinning string spacetime, and the chronology horizon of Misner space and the twin paradox wormhole spacetime.*

Definition 5.7 *Let us now define the hyperbolic region, R_H, to be the region of the spacetime (M, g) containing surfaces $S \in B$, i.e.*

$$R_H = \{x : x \in S \in B\}.$$

The elliptic region will then be the region $M - R_H$.

In the case of the spinning cosmic string we gave arguments for believing it was possible to extend local Cauchy data in some neighbourhood of a point in the hyperbolic region to "global" Cauchy data in the entire hyperbolic region. Furthermore it seems impossible to extend this further to a Cauchy problem which enters part of the elliptic region (at least for generic initial data). We therefore conjecture that this is true in general

Conjecture 5.8 *Given "local" Cauchy data on an achronal spacelike hypersurface in the hyperbolic region, it is possible to extend this local data to "global" Cauchy data in a consistent way to give a well defined Cauchy problem. However for generic "local" data we cannot extend this to "global" data in the elliptic region.*

Note that in the above conjecture we state that it holds for generic initial data. There are special cases of data that can be extended through the essential chronology horizon, as we saw in the examples of Misner space and the spinning string spacetime. Examples of such data is that which propagates along the geodesics whose coordinate time is strictly increase with respect to some affine parameter along the geodesic.

So it seems that in general it is not the Cauchy horizon that is the obstacle in constructing a well defined Cauchy problem, but it is in fact the essential chronology horizon.

As we have already seen the 0^{th}-polarised hypersurface in Grant space, Gott space and the spinning string spacetime shares many common features with the chronology horizon in spacetimes with compactly generated chronology horizons. This leads us to the following conjecture:

Conjecture 5.9 *For spacetimes with compactly generated chronology horizon the essential chronology horizon and chronology horizon coincide. They are distinct for spacetimes with non-compactly chronology horizons.*

5.0.5 Conclusions.

In this thesis we studied the Cauchy problem for the massless scalar wave equation in various causality violating spacetimes. We looked at spacetimes with compactly generated chronology horizons (Misner space and the twins-paradox wormhole spacetime) and spacetimes without compactly generated chronology horizons (Grant space, Gott space and the spinning string spacetime).

The first spacetime we considered was two-dimensional Misner space. This is one of the simplest causality violating spacetimes. Its covering space is Minkowski space with points identified under a boost in the $t - x$ direction (note the origin must be deleted if the covering space is to be a manifold). We solved the wave equation on Misner space given initial data on some spacelike hypersurface preceding the chronology horizon. It was found that generic initial data evolved to a solution of the wave equation that had a divergent stress-energy tensor on the chronology horizon. There are two sets of null geodesics in Misner space. One set of geodesics pass straight through the chronology horizon, while the other set spiral round and round the spacetime never quite reaching the horizon. The divergence is due to the fact that a null ray travelling along the spiralling geodesics becomes infinitely blue shifted as it works its way up to the chronology horizon. However, there are certain choices of initial data that could be evolved in a finite way through the horizon. This corresponds to solutions that propagate only along geodesics that pass straight through the chronology horizon, and hence experience no blue shifting.

Another spacetime with a compactly generated chronology horizon is that of the twin-paradox wormhole spacetime. We reviewed the work done by Kip Thorne et al. for the Cauchy problem in these spacetimes. It was argued that in the geometric optics limit there was in fact a well defined Cauchy problem given initial data on a spacelike hypersurface preceding the chronology horizon. As in Misner space the chronology horizon of this spacetime contains a closed null geodesic, and one might have though this would be a likely place for an instability. However the wormhole acts as a diverging lens, that drives the amplitude of waves to zero as they pass through the wormhole mouths an infinite number of times. If certain parameters are chosen correctly the reduction of the waves amplitude counter acts the blue shifting, leaving the chronology horizon stable. The problem of putting initial data in the non-chronal region was discussed. In the neighbourhood of any point in the non-chronal region one can find a spacelike surface on which to give initial data. However this cannot be extended globally due to a lack of suitable global spacelike surfaces in the non-chronal region on which to put initial data.

We then considered the Cauchy problem in Gott space. By forming a covering space for Gott space we were able to convert the problem to an identical one in Minkowski space. Initial data was posed on a Cauchy surface S_{τ_0} in the chronal region of the physical spacetime. We constructed the data in the covering space by extending data from the physical space via the Gott space isometries. A solution of the wave equation in Minkowski space was constructed using a Kirchoff formula. Unwrapping data from the physical space to the covering space we were able to show that it is possible to construct a solution to the wave equation in Gott space. Using global energy estimates to construct weighted Sobolev spaces we were able to modify results in Hawking and Ellis to prove global existence and uniqueness of solutions to the wave equation in Minkowski space given initial data on surfaces of the form $t^2 - x^2 = \tau_0^2$. Relating function spaces in the covering space to those in the physical space we were able to prove existence and uniqueness of solutions to the wave equation in Gott space. As in Misner space, null geodesics that pass round both strings in Gott space get blue shifted. However null geodesics only spiral round both strings a finite number of times as they approach the chronology horizon in the chronal region. Hence it seem unlikely that initial data given in the chronal region will not evolve in a divergent way as one approaches the chronology horizon.

We next considered the problem of extending solutions to the wave equation of Gott space through the chronology horizon and into the non-chronal region. For simplicity we considered the wave equation on Grant space which has the same acausal properties of Gott space. All solutions to the wave equation on Grant space must be invariant under the Grant space isometries. Going to a suitable set of coordinates these isometries can take the form of a single periodic coordinate. One can now Fourier analyse the wave equation with respect to this periodic coordinate. The resulting two-dimensional reduced wave equation is an equation of mixed type. In one region the equation is hyperbolic, and in the other region it is elliptic. These two regions are separated by a surface of parabolic degeneracy. The wave equation for the spinning cosmic string exhibits identical behaviour when Fourier analysed with respect to its periodic coordinate. On the surface of parabolic degeneracy the orbits of the spacetime's isometries become null. Taking the $n \to 0$ limit of the expression for the n^{th}-polarised hypersurface yields a surface whose location coincides with that of the surface of parabolic degeneracy. For this reason we called the surface of parabolic degeneracy the "0^{th}-polarised hypersurface". The "0^{th}-polarised hypersurface" is a timelike surface that contains closed null curves.

We would like to define the 0^{th}-polarised hypersurface in a way that doesn't depend on either the isometries of the spacetime or the existence of a reduced wave equation. If we consider the Cauchy problem for the reduced wave equation then in the hyperbolic region there exist surfaces that are spacelike with respect to the symbol **h** of the reduced wave equation. These surfaces are partial Cauchy surfaces with respect to **h** as so are suitable surfaces on which to pose initial data for the reduced wave equation. Given a surface $S \in R_H$ one can find a unique solution to the reduced wave equation in $D_2^+(S)$. Such a surface in the spinning string spacetime would be the surface $t = 0$ for $r > \frac{A}{B}$. However when one considers the surface $t = 0$ in the three-dimensional spacetime we see that it is not in fact achronal. However it is achronal within its domain of dependence $D^+(S)$. In order for a null geodesic to intersect $t = 0$ twice it must enter the elliptic region. Return to the Cauchy problem for the reduced wave equation. In the elliptic region there are no surfaces that are spacelike with respect to **h**, and there are no well-posed initial values problems for such elliptic equations. So in the two-dimensional problem we cannot extend our data on a spacelike surface beyond the 0^{th}-polarised hypersurface into the elliptic region. We can again relate this to what happens in the three-dimensional case. We conjectured that as one extends the surface $t = 0$ into the elliptic region it no longer remains achronal within its own domain of dependence, and so the Cauchy problem is not well posed given data on this surface. There are sets of data that can be extended into the elliptic region (as we saw in the last chapter), but it seems likely that these correspond to data that propagate along null geodesics that move against the strings rotation, as such geodesics have a strictly increasing coordinate time t and only have one intersection with spacelike surfaces that wind round the string. This is similar to what happens in Misner space, where data that only travels along one set of geodesics can be extended across the chronology horizon.

We renamed the 0^{th}-polarised hypersurface the "essential chronology horizon". It seems that it is the essential chronology horizon that creates a barrier to there being a well defined Cauchy problem in spacetimes with closed timelike curves. We conjecture that the elliptic region is somehow the source of the causality violation, and that the Cauchy problem on $M - R_E$ is well defined and in fact $M - R_E$ is a perfectly causal spacetime.

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